Name:

Signature for Honor Code:____

"I have neither given nor received aid in completion of this examination."

First Midterm Exam for Physics 212

Professor Henry Greenside
Monday, February 15, 1999

This 75-minute exam is closed book and calculators are not allowed. Please do not hesitate to ask if any questions arise during the exam.

While you do not need to write in complete sentences, your answers must be sufficiently detailed and readable for me to understand what you are doing and why. In questions of the form "True or False and Why", make sure that you indicate true or false and make sure that you justify your answer.

Please provide all work and answers in the space provided for each question. If there is insufficient space, you can use extra paper (available during the exam), which you should label with your name and with the problem number before handing it in with your exam.

Total Score (out of 53): ______

First Midterm Grade: ______

1. **(5 points)** Light going through a y-projector and then through a x-projector is completely blocked. But if you insert a y'-projector (rotated an angle θ from y) between the y- and x-projectors, light can get through all three projectors; the final light intensity is nonzero provided θ is not an integer multiple of $\pi/2$. Explain this result *quantitatively* by using the quantum amplitudes:

$$\langle x|x'\rangle = \cos(\theta), \qquad \langle x|y'\rangle = -\sin(\theta), \qquad \langle y'|y\rangle = \cos(\theta), \qquad \langle x'|y\rangle = \sin(\theta), \tag{1}$$

to calculate the final light intensity $I(\theta)$ through all three projectors in terms of the initial intensity I_0.

2. **(7 points)** A beam of spin-1 particles is prepared in the state

$$|\psi\rangle = \frac{1}{\sqrt{15}}|1,1\rangle - \frac{i}{\sqrt{15}}|1,0\rangle + \frac{2+3i}{\sqrt{15}}|1,-1\rangle, \tag{2}$$

where the kets $|1,m\rangle$ are the three eigenstates of the operator S_z with magnetic quantum numbers $m = 1, 0, -1$. For this superposition, what is the average value of the x-component of the spin angular momentum, $\langle\psi|S_x|\psi\rangle$?

Hint: Use your homework result $S_\pm|s,m\rangle = \hbar\sqrt{s(s+1) - m(m\pm1)}|s, m\pm1\rangle$.

3. A monoenergetic beam of identical neutral atoms is passed through a Stern-Gerlach device whose inhomogeneous magnetic field is oriented in the vertical (z) direction. Eight distinct beams are detected emerging from this device.

   (a) **(3 points)** In units of $\hbar^2$, what is the magnitude squared of the angular momentum of these particles?

   (b) **(3 points)** True or false and why: the observed angular momentum of these particles is due only to orbital motion rather than due to an intrinsic spin.

   (c) **(3 points)** If we label the output beams by integers $1, 2, \cdots, 8$ in descending order from the topmost to bottommost beams, what is the z-component of the angular momentum for the particles in output beam 3? Give your answer in units of $\hbar$.

   (d) **(3 points)** If beam 3 is passed through a second collinear Stern-Gerlach device whose axis of inhomogeneity is rotated 30 degrees from the z direction, how many distinct beams will emerge from this second device?

(e) **(6 points)** For any of the eight output beams, show that the average values of the x- and y-components of the angular momentum are zero, i.e.

$$\langle\psi|L_x|\psi\rangle = \langle\psi|L_y|\psi\rangle = 0, \tag{3}$$

where $|\psi\rangle$ represents the state of a particular output beam.
A hint: Stern-Gerlach machines produce as output eigenstates of L_z, i.e., $L_z|\psi\rangle = (m\hbar)|\psi\rangle$ for some integer m. Try to relate L_x and L_y to L_z through appropriate commutation relations.

(f) **(4 points)** Explain how to measure experimentally the average values $\langle L_x\rangle$ and $\langle L_y\rangle$ for beam 3 so that you can test your prediction Eq. (3).

4. For this next problem, you will want to recall from Physics 211 that the energy levels and wave functions of a particle of mass m in an infinite square well of length W are given by:

$$E_k = \frac{k^2\pi^2\hbar^2}{2mW^2}, \qquad \psi_k(x) = \sqrt{\frac{2}{W}}\sin\left(\frac{k\pi}{W}x\right), \tag{4}$$

while the wave functions of a one-dimensional harmonic oscillator are given by:

$$\psi_n(x) = \left(\frac{1}{n!2^n a\sqrt{\pi}}\right)^{1/2} H_n\left(\frac{x}{a}\right) e^{-x^2/2a^2}, \qquad a^2 = \frac{\hbar}{m\omega_0}, \tag{5}$$

where the Hermite polynomials are given by:

$$H_0 = 1, \qquad H_1 = 2x, \qquad H_{n+1} = 2xH_n - 2nH_{n-1}. \tag{6}$$

Here then is the problem: Consider a three-dimensional potential $V(x, y, z)$ defined by:

$$V(x,y,z) = \begin{cases} \frac{1}{2}m\omega_0^2 z^2, & \text{for } |x| \le L \text{ and } |y| \le L, \\ \infty, & \text{otherwise,} \end{cases} \tag{7}$$

which describes a particle of mass m attached to a spring that can oscillate in the z direction with angular frequency ω_0, but whose motion is otherwise confined by the impenetrable walls of an infinitely long pipe of square cross section.

(a) **(4 points)** What are the bound-state energy levels of this system?

(b) **(3 points)** True or false and why: this potential has only bound states.

(c) **(3 points)** True or false and why: the energy levels of this system can be labeled by quantum numbers n, l, and m corresponding to energy, magnitude-squared of the angular momentum (L^2), and z-component of the angular momentum (L_z).

(d) **(3 points)** What is the wave function $\psi_0(x, y, z)$ of the ground state?

(e) **(3 points)** What is the degeneracy of the four lowest energy levels in the limit $L \gg a$, i.e., the width of the pipe L is much larger than the characteristic size $a = \sqrt{\hbar/(m\omega_0)}$ of the ground state harmonic oscillator wave function?

(f) **(5 points)** For a fixed particle mass m and fixed pipe width L, there is a special value of the oscillator frequency ω_0 such that the degeneracy of many of the energy levels will be as large as possible. What is the value of that frequency and what are the degeneracies of the four lowest energy levels for that special frequency?