Physics 212: Lecture 17

Monday, March 22, 1999
H. S. Greenside

1 Announcements

The second midterm exam will be this Monday, April 5, from 2:15-3:30 pm in Physics 05 (not in the usual LSRC seminar room!). As before, the exam will be closed book and no calculators will be allowed. The exam will cover all material since the last midterm and before this Monday's lecture.

2 Reading

After Thursday's midterm exam, we will discuss Section 5.3 of Griffiths, two simplified models of a "super molecule", i.e., a solid made up of Avogadro's number of nuclei and electrons. We will then cover parts of Chapter 7 on the variational principle, which is a powerful and important strategy for finding the ground state energies of many-particle systems. Here we will also discuss a Monte Carlo method which is the main technique for implementing variational methods on computers and for solving some of the hardest quantum mechanical problems around.

We will then turn to Chapter 6 on perturbation theory, which is one of the few widely applicable ways to gain analytical insight into problems for which full analytical solutions are not possible (which is pretty much all problems except two particle systems like the hydrogen atom). Remaining topics will be chosen depending on the class's interests, e.g., we could discuss tunneling and radioactive decay, some nuclear physics (the shell model), or the semi-classical theory of radiation.

3 Comments on Assignment 5

One problem that most students missed in Assignment 5 was Problem 4.48e of Griffiths (page 172). You were told that an electron in a hydrogen atom occupied the combined spin and position state:

$$R_{21}(r)\left(\frac{1}{\sqrt{3}}Y_1^0\chi_+ + \sqrt{\frac{2}{3}}Y_1^1\chi_-\right), \tag{1}$$

and then you were asked: "If you measured J^2, what values might you get and what is the probability of each?" where $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is the total angular momentum vector.

This was a problem where you needed to use Table 4.7 on page 168 summarizing various Clebsch-Gordon coefficients, following the discussion of Eq. [4.186] in Griffiths, who shows how to use Table 4.7 to express a product state of angular momenta (representing the angular momenta of two separate particles) as a linear combination of total angular momentum states. Now the spherical harmonic Y_1^0 corresponds to the ket $|1,0\rangle$ and the spherical harmonic Y_1^1 corresponds to the ket $|1,1\rangle$, so both spherical harmonics correspond to angular momentum states with $l = 1$. In Eq. (1), you then have two product states, namely $|1,0\rangle\,|1/2,1/2\rangle$ and $|1,1\rangle\,|1/2,-1/2\rangle$.

To express these product states in terms of total angular momentum states, we need to look up and use two rows of the $1 \times 1/2$ subtable in Table 4.7 (this is in the upper left side of the table). The first product state has $m_z = 0$ and $s_z = 1/2$ so we read off from the third row in the subtable:

$$|1,0\rangle\,|1/2,1/2\rangle = \sqrt{\frac{2}{3}}\,|3/2,1/2\rangle - \sqrt{\frac{1}{3}}\,|1/2,1/2\rangle, \tag{2}$$

while the second product state has $m_z = 1$ and $s_z = -1/2$ so the second row of the subtable tells us that:

$$|1,1\rangle\,|1/2,-1/2\rangle = \sqrt{\frac{1}{3}}\,|3/2,1/2\rangle + \sqrt{\frac{2}{3}}\,|1/2,1/2\rangle. \tag{3}$$

If we substitute these two expressions into Eq. (1) and collect terms, we find that the wave function is now expressed in terms of eigenstates of the total angular momentum:

$$\psi = R_{21}(r)\left(\frac{2\sqrt{2}}{3}|3/2,1/2\rangle + \frac{1}{3}|1/2,1/2\rangle\right). \tag{4}$$

From this we see that a measurement of the total angular momentum squared J^2 will give one of two possible values: the value $(3/2)((3/2)+1)\hbar^2 = (15/4)\hbar^2$ with probability $P = |2\sqrt{2}/3|^2 = 8/9$ for $j = 3/2$ or the value $(1/2)((1/2)+1)\hbar^2 = (3/4)\hbar^2$ with probability $|1/3|^2 = 1/9$ for $j = 1/2$.

Both states in Eq. (4) have $j_z = 1/2$ so a measurement of the z-component of the total angular momentum of the state Eq. (1) will give the value $\hbar/2$ with probability $P = 1$.

By the way, the answer to part (g) of this problem (the probability of measuring the position of the electron near the point (r,θ,ϕ) is simply $|\psi|^2$. After looking up the radial function and spherical harmonics in various tables in Griffiths and carrying out some straightforward algebra, one finds the answer:

$$|\psi|^2 = \frac{1}{96\pi a^5} r^2 e^{-r/a},$$

which depends only on the radius, not on the direction in space.

4 Review Questions

The following are miscellaneous questions to help you review for the second midterm:

1. What does "degeneracy" mean in quantum mechanics and why is this an important concept?

2. What is the degeneracy for the $n = 5$ level of hydrogen and justify your answer.

3. What are the total angular momentum values (subscripts) allowed for the following spectroscopic terms:

$$^3S, \quad ^3P, \quad ^4D, \quad ^3F.$$

4. In an earlier lecture, we discussed the vibration-rotation spectrum of a diatomic molecule which consisted of energy levels of the following form:

$$E = \left(n + \frac{1}{2}\right)\hbar\omega_0 + \frac{l(l+1)\hbar^2}{2\mu r_0^2}, \tag{5}$$

where in general the rotational energy spacings are about 1/50th the vibrational energy spacings.

Discuss why or why not Eq. (5) can be applied to a hydrogen atom, which certainly looks like a diatomic molecule consisting of a light electron and heavy proton.

5. What is a "spinor"? Give an example of how a spinor is used in quantum mechanics.

6. What is the Hamiltonian for three non-interacting particles in an infinite square well of width L? Give explicitly the ground state and first excited state wave functions $\psi(x_1, x_2, x_3)$ in the case of distinguishable, fermion, and boson particles. Also give their energies and degeneracies.

7. There is a theorem that says that the ground state wave function of a single-particle Schrodinger equation can not have any nodes (not vanish anywhere in space). Show as a consequence that the ground state wave function must be nondegenerate. Verify that this theorem is correct for the hydrogen atom and the two- and three-dimensional isotropic harmonic oscillators.

8. Consider a bound state $\Psi(t, \mathbf{r})$ of a single particle three-dimensional Schrodinger equation. Derive a condition for this wave function to be normalizable in the form:

$$\int_0^\infty |\Psi|^2 r^\alpha \, dr,$$

i.e., what value of the exponent α should be chosen to guarantee that Ψ is normalizable?

9. True or false and why: a particle which is a bound state of a boson and fermion is itself always a fermion.

10. An external time-dependent electric field can cause the emission or absorption of a photon corresponding to an electron "jumping" from one single-particle three-dimensional stationary state to another stationary state. Such transitions are restricted by so-called "selection rules", which limit the possible changes in quantum numbers describing the initial and final electron states. From a semi-classical theory of radiation interacting with a quantum system, one can show that the selection rules are:

$$\triangle j = 0, \pm 1; \qquad \triangle l = \pm 1; \qquad \triangle m_z = 0, \pm 1; \qquad \triangle n = \text{anything}. \tag{6}$$

Given these rules and ignoring spin, draw a picture of all the possible distinct transitions between the 9 $n = 3$ hydrogen states and the four $n = 2$ hydrogen states.

For the transitions that are allowed, deduce the corresponding energies of an emitted photon and predict what emission lines would be observed in a spectroscope.

11. Over the last fifty years of high energy particle physics, many new particles have been discovered and many of these particles are unstable, with extremely short life times of 10^{-10} sec or less. Since it is not possible to study two or more of these particles as part of some system, explain how one could still determine whether a given particle is a boson or fermion.

12. What is meant by the "exchange energy" between two identical particles and explain the significance of this concept.

13. Consider a Hamiltonian $H = H(p_1, p_2, p_3, x_1, x_2, x_3)$ describing a quantum mechanical system of three identical particles in one space dimension. Also consider the three permutation operators P_{12}, P_{13}, and P_{23} which interchange the positions of any two particles (thus P_{12} applied to some expression switches the indices $1 \leftrightarrow 2$ everywhere).

True or false and why: one can find a stationary state $\psi(x_1, x_2, x_3)$ that is simultaneously an eigenfunction of the four operators H, P_{12}, P_{13}, and P_{23}.

14. What would the periodic table of elements look like if it were based on the quantum states of a three-dimensional isotropic harmonic oscillator instead of the hydrogen atom?

More specifically, consider a three-dimensional isotropic harmonic oscillator with potential $V(r) = (1/2)m\omega_0^2 r^2$. By applying Hund's rules to this system, determine the spectroscopic labeling ${}^{2S+1}L_J$ of the ground state and first six lowest energy states if the oscillator energy levels are populated by non-interacting electrons.

15. If $Y_l^m(\theta,\phi)$ is a spherical harmonic with orbital angular momentum number l and magnetic number m, and if $L_\pm$ denotes the angular momentum lowering and raising operators in spherical coordinates:

$$L_\pm = \frac{\hbar}{i}\left[(-\sin(\phi) \pm i\cos(\phi))\partial_\theta - (\cos(\phi) \pm i\sin(\phi))\cot(\theta)\partial_\phi\right],$$

explain why:

$$L_+ Y_l^l = 0.$$

16. Continuing the previous problem, show that

$$L_\pm Y_l^m = \hbar\sqrt{l(l+1) - m(m\pm 1)}\, Y_l^{m\pm 1}.$$

17. The He nucleus consists of four spin-1/2 particles, two protons and two neutrons. What are all the possible values of the total angular momentum numbers j that might be measured for this nucleus if the orbital angular momentum is always zero? Explain how to test your result experimentally.

18. If the binding energy of a hydrogen atom is -13.6 eV, what is the binding energy of positronium, an atom consisting of an electron and positron? What is the frequency of the Lyman-α line representing a $n=2 \to n=1$ transition?

19. Discuss qualitatively the energy level scheme for a He atom if (a) the electrons were identical bosons and (b) if the electrons were distinguishable particles with the same charge and mass. Assume (inconsistently) that the electrons still have spin 1/2.

20. Consider three non-interacting particles 1, 2, and 3 that each separately are in distinct orthonormal single-particle one-dimensional Schrodinger states $\psi_a(x)$ $\psi_b(x)$, and $\psi_c(x)$ respectively. Write down the normalized three-particle wave function $\psi(x_1,x_2,x_3)$ for the three particles for the following five cases:

    (a) The particles are distinguishable.
    (b) The particles are identical bosons.
    (c) The particles are identical fermions.
    (d) Particle a is a boson and particles b and c are identical fermions.
    (e) Particle a is a fermion and particles b and c are identical bosons.

21. Explain what the Pauli exclusion principle is. Explain how this principle is a consequence of the symmetry of a wave function describing two or more identical fermions.

22. What are the energy levels and wave functions for the $l=0$ states of an isotropic three-dimensional harmonic oscillator?

23. Assuming that the nucleus of a hydrogen atom is a uniform sphere of charge of radius $r = 10^{-13}$ cm, write down a mathematical expression for the probability that an electron in the $1s$ ground state will be observed to lie inside the nucleus. What is the probability for an electron in an excited $2p$ state to be found inside the nucleus?

24. Assuming that the protons and neutrons in a nucleus are non-interacting particles, write down explicitly the ground state wave function of an alpha particle in terms of the single-particle wave functions $\psi_i(\mathbf{x})$ and spin states $\chi_i^\pm$ of its two protons and two neutrons (which we can label as $i = 1,2,3,4$ respectively). An alpha particle has a total angular momentum number $J=0$ and zero orbital angular momentum ($l=0$) for each of its nucleons.