Lecture 11: Physics 212

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1 Reading

Having finished today our discussion of angular momentum and spin, we will now apply our discussion to understanding the quantum mechanics of hydrogen-like atoms. For the next few lectures, please read

1. Sections 4.3.2, 4.1 and 4.2 of Griffiths.
2. Sections 11-1, 11-2 and Chapter 12 of French and Taylor.

2 Appointment for Course Project

Reminder: please make an appointment to discuss your course project.

3 The Neutron and Proton Magnetic Moments

As an interesting application of the addition rule for quantum angular momenta that we have just learned, let's try to understand two experimental facts: the neutron and proton have "funny" magnetic moments μ_n and μ_p that are not simple integer values of the nuclear Bohr magneton $\mu_B = e\hbar/(2m_pc)$:

$$\mu_n = -1.91\mu_B, \qquad \mu_p = 2.79\mu_B. \tag{1}$$

If the neutron and proton were fundamental point particles with no internal structure like the electron, we would expect some simple multiple of μ_B so we might initially guess that numbers like 1.91 and 2.79 come from some kind of internal structure.

The quark theory of baryons indeed says that the neutron and proton are composite particles, each made up of three quarks that are spin-1/2 particles with non-integer charges. The neutron $n = udd$ consists of an up quark u (mass $m_u \approx m_p/3$, charge $e_u = +2/3$) and two down quarks (mass $m_d \approx m_p/3$, charge $e_d = -1/3$) while the proton $p = uud$ consists of two up quarks and a down quark. What is surprising is that we can get a fairly close estimate of the magnetic moments from just this information if we assume that the neutron and proton ground states have no orbital angular momentum and we combine spins appropriately.

Let's work out the details for the neutron and I'll leave for you the similar calculation of the proton magnetic moment. We can measure the magnetic moment of a beam of neutrons with a Stern-Gerlach device, which will split the beam into an upper beam of $|s = 1/2, s_z = 1/2\rangle$ states and a lower beam of $|s = 1/2, s_z = -1/2\rangle$ states. How do we get a net $s_z = 1/2$ spin-1/2 particle from a wave function of three spin-1/2 quarks? One way is to write down a superposition of all possible quark combinations that give the right z-component, e.g.:

$$|1/2, 1/2\rangle_n = c_1|u_\downarrow d_\uparrow d_\uparrow\rangle + c_2|u_\uparrow d_\uparrow d_\downarrow\rangle + c_3|u_\uparrow d_\downarrow d_\uparrow\rangle. \tag{2}$$

Here the notation $u_\uparrow$ or $d_\downarrow$ indicates the z-component of the spin angular momentum is $+\hbar/2$ with a subscript $\uparrow$ or $-\hbar/2$ with subscript $\downarrow$. Eq. (2) contains all the possible up-down combinations of quark spins that have a total z-component of $\hbar/2$.

One could then try to determine the coefficients c_i by using normalization ($c_1^2 + c_2^2 + c_3^2 = 1$ if all coefficients can be chosen real) and by requiring that the right side of Eq. (2) be an eigenstate of the magnitude-squared of the total spin angular momentum $S^2 = (\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3)^2$ with eigenvalue $(1/2)(1/2+1)\hbar^2$ as appropriate for a spin-1/2 particle. Here $\mathbf{S}_i$ denotes the spin operator that operates on the ith particle. Normalization and being an eigenvector of S^2 are only two constraints on three numbers and so is not sufficient to pin down all the coefficients. One needs an extra condition which turns out to be that, if two identical quarks occur in a proton or neutron, they occur symmetrically in the wavefunction. (I won't explain this, it is an experimental fact and associated with the fact that quarks have other quantum numbers called "color".) We must then choose $c_2 = c_3$ to achieve symmetry of the total wave function under interchange of any two identical quarks and then we have enough information.

Challenge: Assuming $c_2 = c_3$, deduce the values of c_1 and c_2 by requiring that Eq. (2) be an eigenvector of S^2.

Alternatively, we can use our summation rules for constructing eigenstates of total angular momentum from angular momentum states of individual particles. From two spin-1/2 down particles, we now know that we can construct a $s = 1$ triplet:

$$|1,1\rangle = |d_\uparrow d_\uparrow\rangle, \qquad |1,0\rangle = \frac{1}{\sqrt{2}}\left(|d_\uparrow d_\downarrow\rangle + |d_\downarrow d_\uparrow\rangle\right), \qquad |1,-1\rangle = |d_\downarrow d_\downarrow\rangle, \tag{3}$$

and a $s = 0$ singlet state:

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|d_\uparrow d_\downarrow\rangle - |d_\downarrow d_\uparrow\rangle\right). \tag{4}$$

Our addition rules tell us further that we can get a $|s = 1/2, s_z = 1/2\rangle$ state by combining the $s = 1$ triplet with the $s = 1/2$ up-quark state (since $s = 1$ and $s = 1/2$ can give a $s = 3/2$ or $s = 1/2$ total angular momentum) or by combining the $s = 0$ singlet with the $s = 1/2$ up-quark state (since $s = 0$ and $s = 1/2$ can give a $s = 1/2$ total angular momentum). Now if we somehow know that identical quarks appear symmetrically in the total wave function of a neutron or baryon, we can reject using the singlet state (which is antisymmetric in both quarks) and so the total wave function must be a superposition of a triplet state of the down quarks and the up-quark. Griffiths' Table 4.7 of Clebsch-Gordon coefficients on page 168 tells us precisely how to combine the wave functions. We look for the sub-table of coefficients combining $s = 1$ with $s = 1/2$ angular momenta (this is the second group of numbers from the top on the far left) and we look for the vertical column with $s = 1/2$ and $s_z = 1/2$ and learn that:

$$|1/2,1/2\rangle = \sqrt{\frac{2}{3}}|1,1\rangle\,|1/2,-1/2\rangle - \frac{1}{\sqrt{3}}|1,0\rangle\,|1/2,1/2\rangle. \tag{5}$$

Substituting from Eq. (3), we now know the coefficients c_i in Eq. (2)

$$|1/2,1/2\rangle = \sqrt{\frac{2}{3}}|u_\downarrow d_\uparrow d_\uparrow\rangle - \frac{1}{\sqrt{6}}\left(|u_\uparrow d_\uparrow d_\downarrow\rangle + |u_\uparrow d_\downarrow d_\uparrow\rangle\right). \tag{6}$$

Again, Eq. (6) contains just two physical assumptions beyond the quark structure: no orbital angular momentum in the ground state and symmetry between identical quarks. If you like, we can take this as one of many hypotheses that one can make about possible wave functions for the neutron ground state.

We are not done yet, the fun is just beginning. Assuming that Eq. (6) describes the ground state wave function of a neutron, we can go ahead and make an experimental prediction by calculating the z-component of the magnetic moment which we know how to measure in principle with a Stern-Gerlach experiment (more realistically, with a Rabi resonance experiment). What is the appropriate operator whose average value will give us this observable? We know that the magnetic moment is related to the spin angular momentum by:

$$\boldsymbol{\mu} = \gamma \mathbf{S}, \tag{7}$$

where γ is the gyromagnetic ratio with value $\gamma = 2(q_e/(2m_e c))$ for a point electron. (Recall that a classical argument gives $q_e/(2m_e c)$ but the electron has a g-factor of 2.) The Dirac relativistic quantum theory for a point spin 1/2 particle suggests that this should hold for a point quark particle also so we will assume that the quark gyromagnetic ratio γ_q obeys a similar relation:

$$\gamma_q = 2\frac{e_q}{2m_q c}. \tag{8}$$

The operator giving the z-component of the neutron magnetic moment would then be:

$$\begin{aligned}\mu_z &= 2\frac{e\hbar}{2m_q c}\left(\frac{2}{3}\sigma_{1z} - \frac{1}{3}\sigma_{2z} - \frac{1}{3}\sigma_{3z}\right)\\ &= 2\left(\frac{m_p}{m_q}\right)\mu_B\left(\frac{2}{3}\sigma_{1z} - \frac{1}{3}\sigma_{2z} - \frac{1}{3}\sigma_{3z},\right),\end{aligned} \tag{9}$$

in terms of the Pauli spin matrices σ_i (recall $\mathbf{S} = (\hbar/2)\boldsymbol{\sigma}$), where we have used the fact that the quark charges are $2e/3$ and $-e/3$ for up and down quarks and I have also used the theoretical/experimental fact that the up and down quarks have almost identical masses, $m_u \approx m_d$, in which case m_q denotes this common value. At this point we have no idea what is a quark mass m_q but a reasonable guess is that it is one-third the neutron or proton mass:

$$m_q \approx \frac{1}{3}m_n \approx \frac{1}{3}m_p \approx 310\text{MeV}. \tag{10}$$

Challenge: Why and how is it possible that the total mass of a composite particle may not be equal to the sum of the masses of its isolated components? In this case, the assumption $m_q \approx m_p/3$ might not be accurate.

Note: A more accurate result from other analysis is that $m_p/m_q \approx 931/336 \approx 2.8$ rather than 3 and further that the down and up quarks have slightly different mass.

We can now do a calculation that is completely familiar to you from our recent lectures and homework, we can calculate the average value:

$$\begin{aligned}\langle 1/2, 1/2|_n\, \mu_z\, |1/2, 1/2\rangle_n = 3\mu_B &\left(\sqrt{\frac{2}{3}}\langle u_\downarrow d_\uparrow d_\uparrow| - \frac{1}{\sqrt{6}}\left(\langle u_\uparrow d_\uparrow d_\downarrow| + \langle u_\uparrow d_\downarrow d_\uparrow|\right)\right)\\ \times &\left(\frac{2}{3}\sigma_{1z} - \frac{1}{3}\left(\sigma_{2z} + \sigma_{3z}\right)\right)\left(\sqrt{\frac{2}{3}}|u_\downarrow d_\uparrow d_\uparrow\rangle - \frac{1}{\sqrt{6}}\left(|u_\uparrow d_\uparrow d_\downarrow\rangle + |u_\uparrow d_\downarrow d_\uparrow\rangle\right)\right).\end{aligned} \tag{11}$$

In your next homework assignment, you will work out the value of this expression and, if successful, you will find that the above expression evaluates to:

$$\langle \mu_z \rangle = -2\,\mu_B, \tag{12}$$

which is to be compared with the experimental value of $\mu_n = -1.91\mu_B$, which is impressive agreement both in sign and magnitude given the simple physical assumptions and approximations so far.

Challenge: Repeat the above calculation for the average z-component of the magnetic moment of a proton $p = uud$ in terms of its quark structure and compare your answer to the experimental value of $\mu_p = 2.79\mu_B$.

Challenge: The effective masses m_d and m_u of the down and up quarks are not identical. Derive formulas for μ_n and μ_p assuming that m_d and m_u are different and deuce these masses from the known experimental values $\mu_n = -1.91\mu_B$ and $\mu_p = 2.79\mu_B$.

4 Concepts to Understand

1. You should know that the simultaneous eigenfunctions of the commuting variables L^2 and L_z where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the *orbital* angular momentum are the normalized spherical harmonics

$$Y_l^m(\theta, \phi) = \epsilon \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} e^{im\phi} P_l^m(\cos(\theta)), \tag{13}$$

where the coefficient ϵ is defined by

$$\epsilon = \begin{cases} (-1)^m & \text{for } m \geq 0 \\ 1 & \text{for } m < 0. \end{cases}, \tag{14}$$

and where the symbol $P_m^l(x)$ denotes the associated Legendre function of x and is defined by:

$$P_l^m(x) = \left(1 - x^2\right)^{|m|/2} \left(\frac{d}{dx}\right)^{|m|} P_l(x), \tag{15}$$

where the Legendre polynomial $P_l(x)$ is itself defined by a recursion relation:

$$P_0 = 1, \qquad P_1 = x, \qquad P_{l+1} = \frac{2l+1}{l+1} x P_l - \frac{l}{l+1} P_{l-1}. \tag{16}$$

The P_l are polynomials of degree l and the associated Legendre functions are polynomials in the quantities x and $\sqrt{1-x^2}$.

With a bit of elbow grease, you should be able to show that the angular momentum operator L^2 in spherical coordinates becomes the following:

$$L^2 = -\hbar^2 \left[\frac{1}{\sin(\theta)} \partial_\theta \left(\sin(\theta) \partial_\theta \cdot \right) + \frac{1}{\sin^2(\theta)} \partial_\phi^2 \right], \tag{17}$$

and we already know that in spherical or cylindrical coordinates,

$$L_z = \frac{\hbar}{i} \partial_\phi. \tag{18}$$

The fact that the spherical harmonics are simultaneous eigenvectors of these operators then leads to the following results which you should memorize and know:

$$L^2 Y_l^m = l(l+1)\hbar^2 Y_l^m; \tag{19}$$

$$L_z Y_l^m = (m\hbar) Y_l^m; \tag{20}$$

$$\langle Y_l^m | Y_{m'}^{l'} \rangle = \int_0^{2\pi} \int_0^{\pi} Y_l^m(\theta, \phi)^* Y_{m'}^{l'}(\theta, \phi) \, \sin(\theta) \, d\theta \, d\phi \tag{21}$$

$$= \delta_{l,l'} \delta_{m,m'}. \tag{22}$$

This last result follows from our favorite theorem for Hermitian operators, that eigenvectors belonging to distinct eigenvalues are orthogonal. (Again you should remember how to prove this using ket notation!)

2. Eq. (17) is important for solving the three-dimensional Schrodinger equation for spherically symmetric potentials $V = V(r)$ since the kinetic energy term has a Laplacian operator ∇^2 which, in spherical coordinates, can be written:

$$\begin{aligned} \nabla^2 &= \partial_x^2 + \partial_y^2 + \partial_z^2 && (23) \\ &= \frac{1}{r^2} \partial_r \left(r^2 \partial_r \cdot \right) + \frac{1}{r^2 \sin(\theta)} \partial_\theta \left(\sin(\theta) \partial_\theta \cdot \right) + \frac{1}{r^2 \sin^2(\theta)} \partial_\phi^2 \\ &= \frac{1}{r^2} \partial_r \left(r^2 \partial_r \cdot \right) - \frac{1}{\hbar^2 r^2} L^2 \\ &= \frac{1}{r} \partial_r^2 (r \cdot) - \frac{1}{\hbar^2 r^2} L^2. && (24) \end{aligned}$$

The angular dependence of the three-dimensional Laplacian in spherical coordinates is exactly the angular momentum operator squared!

3. Note: the spherical harmonics are quite important many areas of applied science and mathematics. E.g., in meteorology and in oceanography, one often represents the three-dimensional time-dependent fields of velocity, pressure, temperature, etc., in terms of finite expansions of spherical harmonics (with radius-dependent coefficients) since this is a natural and economical way to approximate functions on the surface of a sphere. In particular, this approximation avoids the sometimes subtle issue of how to deal with coordinate singularities at $\theta = 0$ and $\theta = \pi$, i.e., at the north and south poles of a sphere.

   The argument goes something like this: since the Y_l^m are the eigenvectors of an observable L^2, they must form a complete set and so any function $f(\theta, \phi)$ must have an expansion in terms of them. If $w(\rho, \theta, \phi)$ denotes some field (e.g., the temperature), we can write:

$$w(\rho, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} w_{lm}(\rho) Y_l^m(\theta, \phi), \tag{25}$$

   since, for any fixed radial distance ρ, we have a function of angles that can be expanded by the completeness of the Y_l^m.

4. Using Eq. (24), we can show the important result that the three-dimensional single-particle Schrodinger equation for a spherically symmetric potential $V = V(r)$ reduces to solving a one-dimensional single-particle Schrodinger equation for some radial function $u(r) = rR(r)$ in a potential

$$V_{1d}(r) = V(r) + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2}, \tag{26}$$

   where the full stationary three-dimensional wave function is

$$\psi(\rho, \theta, \phi) = R(r) Y_l^m(\theta, \phi), \tag{27}$$

   which is simultaneously an eigenfunction of the three operators H, L^2, and L_z.