Lecture 10: Physics 212

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1 Reading

Having finished today our discussion of angular momentum and spin, we will now apply our discussion to understanding the quantum mechanics of hydrogen-like atoms. For the next few lectures, please read

1. Sections 4.3.2, 4.1 and 4.2 of Griffiths.
2. Sections 11-1, 11-2 and Chapter 12 of French and Taylor.

2 Appointment for Course Project

Reminder: please make an appointment to discuss your course project.

3 Concepts to Understand

1. Based on our discussion of the quantum mechanics of a spin 1/2 particle in a strong magnetic field plus a weak transverse rotating magnetic field, you should understand at least qualitatively how NMR works. The ingredients are:
   (a) A resonant Larmor frequency at which there is a large probability of up-spins being flipped to down-spins. Although the mathematics in the handout might be a bit obscure, the physical interpretation is rather clear: this is the frequency of a photon that causes emission or absorption between the two energy levels of a magnetic moment in a constant external magnetic field. It is also the frequency at which the rotating magnetic field is constant and perpendicular to the magnetic moment in the frame of reference rotating with the magnetic moment. This creates a torque that rotates the magnetic moment away from the z-axis, changing its component along the z-axis.
   (b) At a finite temperature T, there is a slightly larger number of spins parallel to the magnetic field (lower energy since $H = -\boldsymbol{\mu} \cdot \mathbf{B}$) than anti-parallel and so there is the possibility of absorbing photons at the resonant frequency. The ratio of up- to down-spin populations is given by the ratio of corresponding Boltzmann factors $\exp(-E_\pm/kT)$, assuming that thermal equilibrium is achieved.
   (c) Finally, there must be another mechanism by which up-spins (excited states) can deexcite to the down-spin state (ground states), so that a steady state of constant power input is possible. Neither of the handouts discuss these mechanisms which are called "spin-lattice relaxation". The relaxation typically occurs by exponential decay $\exp(-t/T_1)$ with a characteristic time T_1 known as the "T_1-relaxation time" or "spin-lattice relaxation time". The time T_1 is another valuable source of information to scientists which supplements the measurement of the gyromagnetic ratio.
2. From Section 4.4.3 of Griffiths, you should understand the main ideas of how quantum angular momentum states add, e.g., the spins of the up and down quarks in a proton, the spins of the electron and proton in a hydrogen atom, the spin and angular momentum of an electron in an excited state of

hydrogen. A full discussion involves some pretty and interesting mathematics (irreducible representations of groups) but would is too specialized and narrow for this semester. Griffiths touches upon the minimal essential ingredients, further details can be found in the Math-Physics references by Shankar and by Landau and Lifshitz.

(a) If one has two angular momentum vectors $\mathbf{L}_1$ and $\mathbf{L}_2$ for two particles, then one can consider a total angular momentum vector operator $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$ and its total z-component $L_z = L_{1z} + L_{2z}$ and try to find simultaneous eigenvectors of L^2 and L_z. The full theory gives a rather simple result: if l_1 is the angular momentum number of one particle and l_2 is the angular momentum number of the other particle, then the two particles together can lead to a total angular momentum number l that can be anywhere in the range:

$$l_1 + l_2, l_1 + l_2 - 1, \cdots, |l_1 - l_2|. \tag{1}$$

For any value of l in this range, the total magnetic quantum number m takes on the usual $2l + 1$ values from $-l$ to l, varying in integer steps.

(b) Eigenvectors of total angular momentum can be constructed from eigenvectors of the separate particles by using a *tensor product* of eigenvectors that is a simple generalization (or analogy) to how we constructed wave functions of three-dimensional potential $V(x, y, z)$ of the form $f(x) + g(y) + h(z)$, i.e., by a product $\psi(x, y, z) = \psi_i(x)\psi_j(y)\psi_k(z)$ where i, j, k are quantum numbers associated with the one-dimensional problems in x, y, and z respectively.

(c) Griffiths works out one of the most common and important examples, namely the total angular momentum arising from two spin-1/2 particles with no orbital angular momentum. By a clever use of "total" ladder operators $L_\pm = L_{1\pm} + L_{2\pm}$, Griffiths shows that two spin-1/2 particles can form a $l = 1$ or $l = 0$ total angular momentum state, which are called traditionally the "triplet" and "singlet" states respectively. As an example, one can write the two distinct states with $m_z = 0$ in terms of single-particle angular momentum states as follows:

$$|l = 1, m = 0\rangle = \frac{1}{\sqrt{2}}\big(|+-\rangle + |-+\rangle\big) \tag{2}$$

$$|l = 0, m = 0\rangle = \frac{1}{\sqrt{2}}\big(|+-\rangle - |-+\rangle\big). \tag{3}$$

The notation $|+-\rangle$ is an abbreviation for the product:

$$|+-\rangle = |s_1 = 1/2, s_{1z} = 1/2\rangle\, |s_2 = 1/2, s_{2z} = -1/2\rangle, \tag{4}$$

and it is understood that an operator of the form S_{1i} or S_{2i} with $i = x, y, z, \pm$ acts on the product by acting only on the state with the same number. Thus:

$$(S_{1z} + S_{2z})|s_1 = 1/2, s_{1z} = 1/2\rangle\, |s_2 = 1/2, s_{2z} = -1/2\rangle \tag{5}$$

$$= (S_{1z}|s_1 = 1/2, s_{1z} = 1/2\rangle)\, |s_2 = 1/2, s_{2z} = -1/2\rangle \tag{6}$$

$$+ |s_1 = 1/2, s_{1z} = 1/2\rangle\, (S_{2z}|s_2 = 1/2, s_{2z} = -1/2\rangle) \tag{7}$$

$$= \left(\frac{\hbar}{2} - \frac{\hbar}{2}\right) \cdots \tag{8}$$

$$= 0. \tag{9}$$

(d) The singlet state of two spins indicates that the spins of two particles are highly correlated. This leads to one version of the famous Einstein-Podolsky-Rosen paradox (or "EPR paradox) which Einstein and collaborators invented to show that something was seriously wrong with quantum mechanics. This is discussed briefly in Section A.1 of Griffiths (pages 375-376) which you should read. The idea is for a particle with initial zero total angular momentum, (e.g., a pi meson π) to decay spontaneously into two particles of opposite spin (e.g., an electron and positron). One then lets the two decay pieces travel a large distance away from each other and then carry out a measurement of the z-component of the spin along some arbitrary direction z for one of the particles, say the electron. Now if you repeat this measurement over and over again, one will get different values ($+\hbar/2$, $-\hbar/2$, $-\hbar/2$, etc) each time since the decay particles are not in an eigenstate of S_z. But as soon as the z-component is known for one particle, it must be known for the other particle by conservation of angular momentum and so a measurement on one particle fixes the value of S_z for the other particle instantly, no matter how far away the two particles are. This *non-locality* of quantum mechanics, that a measurement in one space-time point can instantly affect a measurement outside a causal light cone, deeply bothered Einstein (and should bother most other people). Experiment tests however show that quantum mechanics is correct and nonlocal.

3. You should know that the simultaneous eigenfunctions of the commuting variables L^2 and L_z where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the *orbital* angular momentum are the normalized spherical harmonics

$$Y_l^m(\theta, \phi) = \epsilon \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} e^{im\phi} P_l^m(\cos(\theta)), \tag{10}$$

where the coefficient ϵ is defined by

$$\epsilon = \begin{cases} (-1)^m & \text{for } m \geq 0 \\ 1 & \text{for } m < 0. \end{cases}, \tag{11}$$

and where the symbol $P_m^l(x)$ denotes the associated Legendre function of x and is defined by:

$$P_l^m(x) = \left(1 - x^2\right)^{|m|/2} \left(\frac{d}{dx}\right)^{|m|} P_l(x), \tag{12}$$

where the Legendre polynomial $P_l(x)$ is itself defined by a recursion relation:

$$P_0 = 1, \qquad P_1 = x, \qquad P_{l+1} = \frac{2l+1}{l+1} x P_l - \frac{l}{l+1} P_{l-1}. \tag{13}$$

The P_l are polynomials of degree l and the associated Legendre functions are polynomials in the quantities x and $\sqrt{1-x^2}$.

With a bit of elbow grease, you should be able to show that the angular momentum operator L^2 in polar coordinates becomes the following:

$$L^2 = -\hbar^2 \left[\frac{1}{\sin(\theta)} \partial_\theta \left(\sin(\theta) \partial_\theta \cdot \right) + \frac{1}{\sin^2(\theta)} \partial_\phi^2 \right], \tag{14}$$

and we already know that in spherical or cylindrical coordinates,

$$L_z = \frac{\hbar}{i} \partial_\phi. \tag{15}$$

The fact that the spherical harmonics are simultaneous eigenvectors of these operators then leads to the following results which you should memorize and know:

$$\begin{aligned}
L^2 Y_m^l &= l(l+1)\hbar^2 Y_m^l; && (16)\\
L_z Y_m^l &= (m\hbar) Y_m^l; && (17)\\
\langle Y_m^l | Y_{m'}^{l'} \rangle &= \int_0^{2\pi} \int_0^{\pi} Y_m^l(\theta, \phi)^* Y_{m'}^{l'}(\theta, \phi) \, \sin(\theta) \, d\theta \, d\phi && (18)\\
&= \delta_{l,l'} \delta_{m,m'}. && (19)
\end{aligned}$$

This last result follows from our favorite theorem for Hermitian operators, that eigenvectors belonging to distinct eigenvalues are orthogonal. (Again you should remember how to prove this using ket notation!)

4. Eq. (14) is important for solving the three-dimensional Schrodinger equation for spherically symmetric potentials $V = V(r)$ since the kinetic energy term has a Laplacian operator ∇^2 which, in spherical coordinates, can be written:

$$
\begin{aligned}
\nabla^2 &= \partial_x^2 + \partial_y^2 + \partial_z^2 & (20)\\
&= \frac{1}{r^2}\partial_r\left(r^2\partial_r\cdot\right) + \frac{1}{r^2\sin(\theta)}\partial_\theta\left(\sin(\theta)\partial_\theta\cdot\right) + \frac{1}{r^2\sin^2(\theta)}\partial_\phi^2 & (21)\\
&= \frac{1}{r^2}\partial_r\left(r^2\partial_r\cdot\right) - \frac{1}{\hbar^2 r^2}L^2. & (22)
\end{aligned}
$$

The angular dependence of the three-dimensional Laplacian in spherical coordinates is exactly the angular momentum operator squared!

5. Note: the spherical harmonics are important in meteorology, for simulating the weather numerically. Since the Y_m^l are the eigenvectors of an observable L^2, they must form a complete set and any function $f(\theta,\phi)$ must have an expansion in terms of them. Thus if $w(\rho,\theta,\phi)$ denotes the pressure or temperature or velocity component field of the air in the earth's atmosphere, we can write:

$$w(\rho,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} w_{lm}(\rho)Y_l^m(\theta,\phi), \tag{23}$$

since, for any fixed radial distance ρ, we have a function of angles that can be expanded by the completeness of the Y_m^l. Such expansions provide economical and powerful ways to solve the partial differential equations governing the weather, the oceans, plate tectonics, star dynamics, etc.