Lecture 8: Physics 212

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H. S. Greenside

1 Reading

1. You should finish reading the two brief handouts on nuclear magnetic resonance (NMR), an important illustration and application of spin 1/2 particles.

2. Please finish reading Griffiths Sections 4.3 and 4.4. We will finish our discussion of spin with Section 4.4.3 of Griffiths (how to add angular momenta) and then we will backtrack a bit and apply our discussion of angular momentum to several single-electron three-dimensional problems (Griffiths Sections 4.1 and 4.2, French and Taylor Sections 5-1 through 5-5 and Chapter 12). We will finally discuss the hydrogen atom but with the full power of quantum mechanics behind us.

   If you would like to read ahead, the next topic we will be discussing in the course is the quantum mechanics of many-particle systems, Chapter 5 of Griffiths and Chapter 13 of French and Taylor. Together with the Schrodinger equation and spin, the symmetry of many-particle wave functions will be the last fundamental physics that we need to know before turning to many applications.

2 Thought Questions

1. Consider an $x-y$ analyzer and an $x'-y'$ analyzer, the latter rotated an angle θ from the y-axis. Using the French and Taylor convention for various amplitudes, explain why the two circularly polarized states:
$$|R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle), \qquad |R'\rangle = \frac{1}{\sqrt{2}}(|x'\rangle + i|y'\rangle), \tag{1}$$
   are physically equivalent.

2. Show that the operator L_z in spherical coordinates ρ, θ, ϕ has the same form $(\hbar/i)\partial_\phi$ as it does in cylindrical coordinates.

3. Why is the magnetic moment of a nucleus so much smaller than the magnetic moment of an electron?

4. If $|\psi\rangle$ is an eigenstate of L_z so that $L_z|\psi\rangle = (m\hbar)\psi\rangle$, show that the average value $\langle L_{z'}\rangle$ of the angular momentum about a z'-axis that is rotated an angle θ from z is given by $m\cos(\theta)$.

3 Concepts to Understand

1. In reading quantum mechanics books and papers, you will come across the concept of a "good quantum number" or a "bad quantum number", usually referring to the orbital and magnetic quantum numbers l and m. The good and bad refers to whether the corresponding operators $\mathbf{L}^2$ and L_z commute with the Hamiltonian of a given system. If so, one can classify all stationary states by energy, l, and m and so l and m are "good" or useful for understanding the quantum states. If either L^2 or L_z do not commute with H, then one can generally not find kets that are simultaneous eigenfunctions of H and of these operators and the corresponding quantum number l or m are bad.

   One can usually identify good or bad quantum numbers immediately by the symmetry of the potential $V(x,y,z)$. E.g., if V is spherically symmetric, $V = V(r)$, $[L^2, V] = 0$, and l is a good quantum number. If V has axial symmetry ($V = V(r, z)$ in some cylindrical coordinate system), then m is a good quantum number but l is not. As an example, the quantum states of a single particle in a three-dimensional box with infinite high potential walls (i.e, $V = 0$ for $|x_i| \le L_i$, $i = 1, 2, 3$) lacks both spherical and axial symmetry (continuous rotations of a cube won't transform into the same cube) and so l and m are bad quantum numbers.

2. Section 4.4.1 of Griffiths showed how to find the 2×2 matrix representation of any spin operator, e.g., S_z or S_x by examining its effect on the spin-up and spin-down basis elements $|\chi_+\rangle$ and $\chi_-\rangle$. You should appreciate that one can calculate various averages such as $\langle S_x \rangle$ without knowing the explicit representation for an operator like S_x by simply using orthogonality of the spin basis elements, by expressing $S_x = (S_+ + S_-)/2$ and $S_y = (S_+ - S_-)/(2i)$ in terms of ladder operators, and then using the important relation:
$$S_\pm |s, m\rangle = \hbar \sqrt{s(s+1) - m(m \pm 1)}\,|s, m \pm 1\rangle. \tag{2}$$

3. You should understand how a spin vector interacts with a magnetic field $\mathbf{B}$ through the Hamiltonian $H = -\mu \cdot \mathbf{B} = -\gamma \mathbf{B} \cdot \mathbf{S}$ and the implications of this Hamiltonian for measuring various spin properties as a function of time.

4. You should appreciate that the average angular momentum vectors $\langle \psi | \mathbf{L} | \psi \rangle$ often behave like classical vectors when a magnetic field is turned on, which is quite helpful when trying to anticipate the behavior of a quantum mechanical system. E.g., for a bound particle moving in a potential $V(\mathbf{r})$, the time derivative of the average angular momentum is given by the average of the quantum mechanical torque operator $\mathbf{N}$ as follows:
$$\frac{d}{dt} \langle \mathbf{L} \rangle = \langle \mathbf{N} \rangle, \tag{3}$$
where the torque operator is given by:
$$\mathbf{N} = \mathbf{r} \times (-\nabla V). \tag{4}$$
The quantity $-\nabla V$ is classically just the force acting on a particle.

   Let me show you how one would start to derive Eq. (3). One of your midterm questions from Physics 211 asked you to derive the following formula for the time derivative of a general operator $Q(t)$:
$$\frac{d}{dt} \langle Q \rangle = \frac{i}{\hbar} \langle [H, Q] \rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle, \tag{5}$$
where H is the Hamiltonian of the system. (This is Eq. [3.148] on page 113 of the Griffiths book). In this formula, let's choose
$$Q = \mathbf{L} = \mathbf{r} \times \mathbf{P}. \tag{6}$$
This is time independent so $\partial_t Q = 0$ and Eq. (5) tells us that
$$\frac{d}{dt} \langle \mathbf{L} \rangle = \frac{i}{\hbar} \langle [H, \mathbf{L}] \rangle, \tag{7}$$
so the time dependence of the average angular momentum is determined by the commutator of the Hamiltonian $H = \mathbf{P}^2/2m + V$ with $\mathbf{L}$.

   Let's look at the x-component of the vector relation Eq. (7), since we can deduce the other components by an appropriate cyclic permutations of symbols. We have:
$$[H, L_x] = [H, yP_z - zP_y] = y[V, P_z] - z[V, P_y], \tag{8}$$
using $[A, BC] = [A, B]C + B[A, C]$, $[P_y^2, y] = -2i\hbar P_y$, and $[\mathbf{P}^2, P_i] = 0$ for $i = x, y, z$. But in an earlier homework exercise, you showed that
$$[f(x), P_x] = i\hbar \partial_x f, \tag{9}$$
and generalizing this to several variables we see that:
$$[V, P_z] = i\hbar \partial_z V, \qquad [V, P_y] = i\hbar \partial_y V, \tag{10}$$

and so:

$$[H, L_x] = y[V, P_z] - z[V, P_y] = i\hbar\,(y\partial_z V - z\partial_y V) = i\hbar\big[\mathbf{r}\times(-\nabla V)\big]_x, \tag{11}$$

which shows that:

$$\frac{d}{dt}\langle L_x\rangle = \langle[\mathbf{r}\times(-\nabla V)]_x\rangle, \tag{12}$$

and so Eq. (3) follows.

A neat consequence of Eq. (3) is a statement of conservation of angular momentum in the absence of an external torque. In particular, if one has a central potential that is a function of radius only, $V(x,y,z) = V(r)$, then ∇V is parallel to $\mathbf{r}$, the average $\langle\mathbf{N}\rangle$ of the torque operator vanishes, and we have $d\langle\mathbf{L}\rangle/\mathbf{dt} = \mathbf{0}$.

Eq. (3) also tells us that the average of a spin vector $\langle\mathbf{S}\rangle$ precesses in a constant homogeneous magnetic field $\mathbf{B}$ since a magnetic moment $\mu = \gamma\mathbf{S}$ feels a torque $\mathbf{N} = \mu\times\mathbf{B}$. To get a sense of the order of magnitude, the angular frequency of rotation of a spin $1/2$ electron in a strong magnetic field of 1 Tesla (10^4 Gauss) would be:

$$\omega = \gamma B_0 = \frac{e}{cm_e}10^4 = \frac{4.8\times10^{-10}}{(3\times10^{10})(9.1\times10^{-28})}10^4 \approx 2\times10^{11}\,\mathrm{sec}^{-1}, \tag{13}$$

i.e., a frequency in the 100 GHz range. The magnetic moment of a nucleus would be down by a factor of about 2000 and so the Larmor frequency in the same field would be about 100 MHz.

5. Griffiths' quantum mechanical discussion of the Stern-Gerlach device is flawed. In Eq. [4.171], Griffiths ignores the fact that as soon as a particle enters the inhomogeneous field, it starts to accelerate up or down depending on the z-component of its magnetic moment. His Figure 4.8 on page 163 is also flawed in that he shows the beams splitting upon exiting the device while, in fact the splitting occurs immediately on the left and increases from left to right. As a consequence, the Hamiltonian in Eq. [4.171] on page 163 should have a z dependence that itself is a function of time, representing the particles moving steadily up or down in the in homogeneous field. Since Griffiths gets the final answer right, there must be some cancellation from this time-dependence. Still, the discussion is not accurate and so confusing.

6. Nuclear magnetic resonance or NMR is an ingenious and valuable way to explore the magnetic properties of atomic nuclei. One way to think about it is to first consider that a nucleus with a magnetic moment with spin number s has $2s+1$ energy levels in the presence of a magnetic field, representing the possible eigenstates of S_z. One could then imagine photons being absorbed or emitted with energies corresponding to the differences in energy levels $\triangle E = -\gamma\hbar B_0\triangle m$ and a study of these photons would give information about the magnetic moment. (Conservation of angular momentum requires that $\triangle m = \pm1$ when a photon is emitted or absorbed so, in fact, transitions can only occur between adjacent levels.) In the 1950's, Edward Purcell and independently Felix Bloch hit upon the idea (and demonstrated the idea in the laboratory) of using an oscillatory magnetic field, perpendicular to a given large magnetic field $B_0\mathbf{z}$, to stimulate transitions between these levels and so measure the tiny and otherwise inaccessible nuclear magnetic moments.

   The quantum dynamics of a spin $1/2$ particle in a field of the form

$$\mathbf{B} = (b\cos(\omega t), b\sin(\omega t), B_0), \qquad B_0 \gg b, \tag{14}$$

   provides the foundation for NMR. (Note: NMR is used heavily in the medical community but has been renamed "magnetic resonance" because many patients were scared by the "nuclear" part of this name, mistakingly thinking that radioactive materials were involved).