Lecture 7: Physics 212

Thursday, February 11, 1999
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1 No Lecture Notes for 2/8/99

Note: There were no lecture notes for Monday's lecture on February 8, 1999.

2 First Midterm This Monday

The first of two midterms will be this Monday at 2:15pm in Physics 05 (*not* in the LSRC seminar room). The exam will cover all material up to today. As before, the exam will be closed book and calculators are not allowed.

3 Reading

1. You should read the two brief handouts on nuclear magnetic resonance (NMR), an extremely important application of quantum spin. We will discuss this in class next Thursday (a week from today) and also visit Professor Al Johnson's lab that uses NMR for high-resolution medical imaging.
2. Please finish reading Griffiths Sections 4.3 and 4.4. We will finish our discussion of spin with Section 4.4.3 of Griffiths (how to add angular momenta) and then we will backtrack a bit and apply our discussion of angular momentum to several single-electron three-dimensional problems (Griffiths Sections 4.1 and 4.2, French and Taylor Sections 5-1 through 5-5 and Chapter 12). We will finally discuss the hydrogen atom but with the full power of quantum mechanics behind us.

   If you would like to read ahead, the next topic we will be discussing in the course is the quantum mechanics of many-particle systems, Chapter 5 of Griffiths and Chapter 13 of French and Taylor. Together with the Schrodinger equation and spin, the symmetry of many-particle wavefunctions will be the last fundamental physics that we need to know before turning to many applications.

4 Comment On Assignment 2

Most of the class had trouble solving Problem 10-5 on page 465 of French and Taylor, where you were supposed to work out the beam profile for a classical Stern-Gerlach device. The problem had two parts, the first assuming that you had a monoenergetic beam (all particles have exactly the same velocity), the second assuming that there was a distribution of velocities.

The challenge in the first part was to calculate the probability distribution function $P(\mu_z)$ for the z-component of the magnetic moments of atoms in the beam, given that the magnetic moments come out of the Stern-Gerlach oven with all possible angles, equally distributed. From Eq. (10-5) on page 431, we see that, for a fixed velocity v, we only need to know how the quantity μ_z is distributed since that is the only quantity that causes the atoms to be spread apart in the z-direction.

The desired pdf $P(\mu_z)$ is obtained by a change of variables as follows. First, we make the assumption that the magnetic moments all lie in a plane, say the $x-z$ plane if x is the direction of propagation. (I will let you work out the slightly more general case if the orientation of the magnetic moments vary randomly and uniformly in the angles θ and ϕ of a spherical coordinate system moving with the center of mass of each atom.) We then assume that each magnetic moment vector $\mu = \mu(\cos(\theta), \sin(\theta))$ lies in this $x-z$ plane with an orientation θ that is uniformly and randomly distributed in the interval $[0, 2\pi]$. This means that if we choose an atom at random in the beam, the probability $P(\theta)$ of its magnetic moment having an angle in the interval $[\theta, \theta + d\theta]$ is given by:

$$P(\theta)\, d\theta = \frac{1}{2\pi} d\theta. \tag{1}$$

Now we know that the z-component of the magnetic moment is related to the angle θ by the following:

$$\mu_z = \mu \cos(\theta), \tag{2}$$

which tells us how an infinitesimal change in θ is related to an infinitesimal change in μ_z:

$$d\mu_z = -\mu \sin(\theta)\, d\theta. \tag{3}$$

We then simply change variables from θ to μ_z in Eq. (1) to get the pdf for μ_z:

$$P(\theta)\, d\theta = \frac{1}{2\pi} d\theta = \frac{1}{2\pi} \frac{d\mu_z}{\mu \sin(\theta)} = \frac{1}{2\pi} \frac{d\mu_z}{\sqrt{\mu^2 - \mu_z^2}}, \tag{4}$$

where I used the fact that

$$\sin(\theta) = \sqrt{1 - \cos(\theta)^2} = \sqrt{1 - (\mu_z/\mu)^2}. \tag{5}$$

We conclude that the probability distribution function for μ_z is given by:

$$P(\mu_z) = \frac{1}{2\pi} \frac{1}{\sqrt{\mu^2 - \mu_z^2}}. \tag{6}$$

This makes the interesting prediction that the beam profile will be continuous but not uniform; the profile of atoms on the glass detector will be much denser near the top and bottom parts of the interval where the atoms are deposited after exiting the Stern-Gerlach device. A picture of this distribution is show in Fig 3-6(c) on page 122 of French and Taylor. (Can you understand why the same distribution shows up in the simple pendulum?)

One could also have discovered the pdf Eq. (6) by using a so-called Monte-Carlo technique, in which you use a computer to generate randomly chosen samples of angles and then you empirically calculate a histogram of resulting magnetic moment components μ_z. Monte-Carlo methods are enormously powerful and useful since they can pretty much solve any problem of any complexity given enough computer time. Their drawback is that they never lead to a concise precise statement like Eq. (6)

A Monte-Carlo investigation would go something like this. First, choose some large integer N which is the number of samples that you will generate and analyze, e.g., $N = 10,000$ (you can expect statistical errors to then be of order $1/\sqrt{N}$.) Next, use a computer to generate N random numbers in the interval $[0, 2\pi)$, representing a sample of random magnetic moments coming out of the Stern-Gerlach oven. With the Mathematica program, one can do this with a single line of code

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angles = Table[ Random[Real,{0,2Pi}], {N} ]
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and the symbol "angles" would be a list of N numbers. Note: There are lots of interesting number theory and many subtleties about how to generate high quality uniformly-distributed random numbers in some interval but you can basically considered this a solved problem. The Mathematica function Random incorporates (and hides) the details of this extensive research.

Given N random uniformly-distributed numbers θ in the interval $[0, 2\pi]$ (this is the Monte-Carlo or random part of the solution), you can then calculate N z-components of a magnetic moment $\mu_z = \mu \cos(\theta)$ which we can assume all to be of unit length, $\mu = 1$. In Mathematica, this is also done trivially:

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muz = Cos[ angles ]
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which gives a new list of N numbers stored in the symbol "muz". Mathematica knows how to distribute a function like cos over each element of a list so you don't need to use any loops as in C or Fortran.

We can then break the interval $[-1, 1]$ representing the range of possible z-components of μ into some finite number M of equally spaced intervals ("bins" in the language of statisticians), say $M = 30$, and count how many numbers from the list "muz" fall into any given bin. After going through all the numbers in muz, we will have a discrete approximation to the unknown distribution $P(\mu_z)$ that we can make more and more accurate by first increasing the number of bins, M, and then increasing the number of samples N. (You need to increase N with increased M else most bins won't have anything inside of them.) The true continuous distribution is recovered in the limits $M, N \to \infty$ although this can never be achieved by a computer program.

5 Thought Questions

1. Can one measure the z-component of a particle's spin and its momentum vector $\mathbf{P}$ simultaneously? Can one measure the z-component of a particle's spin and it position vector $\mathbf{X}$ simultaneously? If so, sketch an apparatus that would achieve these simultaneous measurements.

2. Consider a beam of silver atoms (which are spin-1/2 particles) coming out of a Stern-Gerlach experiment oriented in the z-direction. If we block the spin-down beam and let the spin-up beam enter into a Stern-Gerlach experiment with its field inhomogeneity rotated at an angle θ from the vertical (call this the z' axis), what is the probability of observing angular momentum $\hbar/2$ along z'? Is this completely analogous to z-polarized light entering a z'-polarized filter, is the intensity proportional to $\cos(\theta)^2$?

3. A spinor is a complex-valued two-component vector that describes a three-dimensional vector (a spin-1/2 angular momentum vector). To understand this fact, observe that an arbitrary normalized spinor can be parameterized in terms of two angles α and β as follows:

$$|\chi\rangle = e^{i\delta} \begin{pmatrix} e^{i\alpha/2} \cos(\beta/2) \\ e^{-i\alpha/2} \sin(\beta/2) \end{pmatrix}, \tag{7}$$

since one can have an arbitrary overall phase factor $e^{\imath\delta}$, one can have an arbitrary relative phase factor $e^{i\alpha}$ (here shared equally between the two components), and, as a normalized vector, the magnitude squared of the components must add to one.

For this arbitrary normalized spinor, calculate the average spin vector

$$\langle \mathbf{S} \rangle = \langle \chi | \mathbf{S} \chi \rangle = \big(\langle S_x \rangle, \langle S_y \rangle, \langle S_z \rangle \big), \tag{8}$$

think about what your answer means, and then explain the physical meaning of the angles α and β. If you rotate the three-dimensional coordinate system by an angle of 2π around the z-axis, does the spinor χ remain unchanged?

6 Concepts to Understand

1. You should understand how to manipulate spinors and spin operators, you should be able to calculate average value of any spin-related observable.

2. You should understand how a spin vector interacts with a magnetic field $\mathbf{B}$ through the Hamiltonian $H = -\mu \cdot \mathbf{B}$ and the implications of this Hamiltonian for measuring various spin properties as a function of time.