# Lecture 5: Physics 212

Monday, February 1, 1999
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## 1 Thursday Class Moved to Wednesday

This week's Thursday class is moved to Wednesday, February 3, from 12:30-1:45 pm, in the LSRC seminar room.

## 2 Reading

1. Chapters 10 and 11 of French and Taylor.
2. Sections 4.3 and 4.4 of Griffiths (angular momentum and spin).
3. Browse through the article "Electrons in artificial atoms" by R. C. Ashoori (Nature **379**:413-419 (1996) to identify the key points (but don't read in detail unless you really want to).

## 3 Concepts to Understand

1. You should understand the principles of the Stern-Gerlach experiment and how it sorts a beam of neutral particles according to the component of magnetic moment along the axis of field inhomogeneity (arbitrarily and usually denoted as the $z$ axis). You should also appreciate some of the experimental details, e.g., the clever insight of how to create an inhomogeneous magnetic field and the size and magnitude of various quantities. It is unfortunate that most physics textbooks (undergraduate and graduate) emphasize theory overwhelmingly over experiment since experiment is absolutely essential in testing theory and often involves many ingenious insights of its own.

   You should understand the analogy between a Stern-Gerlach device and a polarization analyzer: both decompose a beam into orthogonal and complete states so one can calculate projection probabilities and amplitudes with suitably chosen experiments. As an example, if the "up"-beam from a Stern-Gerlach device is fed into another Stern-Gerlach device but with the inhomogeneous magnetic field rotated at an angle $\theta$ from the vertical, what kind of beams will come out of the second device?

2. You should understand what sets the magnitude of atomic magnetic moments. In particular, why are magnetic moments of nuclei so much smaller than magnetic moments of the electrons orbiting the nuclei?

3. You should recall the concept of *degeneracy* in quantum mechanics, which means there are two or more linearly independent solutions to the time-independent Schrodinger equation. One then says that a particular energy level is "degenerate" and its degree of degeneracy is the number of independent solutions of the same energy.

   You should recall our analysis from Physics 211 which showed that all bound states of the *one*-dimensional time-independent single-particle Schrodinger equation are non-degenerate. The idea was to assume the opposite, that there were two linearly independent *normalized* solutions $\psi_1(x)$ and $\psi_2(x)$, and write down the Schrodinger equations for these two solutions:

$$\frac{-\hbar^2}{2m}\partial_x^2\psi_i + V(x)\psi_i = E\psi_i, \qquad i = 1, 2. \tag{1}$$

One multiplied the first equation by $\psi_2$, the second equation by $\psi_1$, subtracted the two equations to kill a few identical terms, to obtain:

$$0 = \partial_x \left( \psi_2 \partial_x \psi_1 - \psi_1 \partial_x \psi_2 \right). \tag{2}$$

So the quantity being differentiated must be a constant and that constant must be zero since $\psi_i$ and $\partial_x \psi_i$ both vanish at $\pm\infty$ since they are normalized. We conclude that $\partial_x \psi_1 / \psi_1 = \partial_x \psi_2 / \psi_2$ which implies that one wavefunction is a constant times the other, i.e., they are linearly dependent, a contradiction.

Why is the concept of degeneracy important? As French and Taylor show with their discussion in Chapter 10 of the two-dimensional isotropic quantum harmonic oscillator (also in Section 5-4 for a 3d box), the bound states of two- and three-dimensional quantum systems that have some kind of symmetry (here isotropy under rotation) are typically degenerate. It then turns out (and we shall study this later in the semester when we study quantum perturbation theory) that perturbations of the system, e.g., the addition of a small external electric or magnetic field or the presence of a nearby particle, "break" the degeneracy in the sense of making all energy levels have slightly different energies. Studying how previously degenerate levels are spread apart by external perturbations can provide valuable insight about a quantum system. As one example, many spectroscopic lines are doublets when examined under high resolution (sodium metal has a famous such doublet, see Figure 10-1 of French and Taylor) and the origin of this splitting turns out to be an interesting relativistic effect (spin-orbit coupling).

**Challenge:** Consider a normalized wavefunction $\psi(x, y, z)$ satisfying a single-particle three-dimensional time-independent Schrodinger equation. Show that if $\psi$ is everywhere nonzero, then it must be nondegenerate.

4. Review from Chapter 3 of Griffiths the significance of two observables (Hermitian operators with complete eigenvectors) commuting: one can find a set of kets which are simultaneously eigenvectors of mutually commuting operators.

   You should then understand and appreciate French and Taylor's discussion of the two-dimensional isotropic harmonic oscillator (Section 10-8) as an example where one can construct wave functions that are simultaneously eigenfunctions of energy (the Hamiltonian) and of the $z$ component of the angular momentum. One does this by taking advantage of the degeneracy of the energy levels. In the $|n_x, n_y\rangle$ representation, one can form linear superpositions that are simultaneously eigenvectors of $H$ and of $L_z$. The final result (see Table 10-2 on page 462 of French and Taylor) lead to a surprising result: if $n$ denotes the energy quantum level of a particular stationary state of the 2d harmonic oscillator, the magnetic quantum number varies in steps of 2 from its largest to smallest values:

   $$m = -n, -(n-2), \ldots, n-4, n-2, n. \tag{3}$$

   This is unusual (physicists call it an "accident" while mathematicians would likely call it a "pathology"). For most systems, $m$ takes on all integer values between its max and min values.

   The construction of simultaneous eigenkets of $H$ and $L_z$ for the two-dimensional isotropic harmonic oscillator provides valuable insight into the quantum levels of quantum dots, which are two-dimensional circular objects made from VLSI technology, with an approximately parabolic potential.

   We will see in a few lectures that the three-dimensional isotropic harmonic oscillator provides valuable insights into the quantum structure of nuclei so, again, it is important to understand the degenerate structure of the stationary states and how to obtain simultaneous eigenkets of $H$ and $L_z$.

5. Know how to derive and use various commutation relations involving the operator vector angular momentum $\mathbf{L}$:

   $$\mathbf{L} \times \mathbf{L} = i\hbar \mathbf{L}, \qquad [L^2, \mathbf{L}] = 0,, \qquad [L^2, f(r)] = 0. \tag{4}$$

6. Review your chain rule so that you understand why the $z$-component of the angular momentum operator

$$L_z = xp_y - yp_z, \tag{5}$$

turns into

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\theta}, \tag{6}$$

when one changes coordinates from Cartesian to cylindrical, $(x, y, z) \to (r, \theta, z)$, or from Cartesian to spherical, $(x, y, z) \to (r, \theta, \phi)$.

More generally, you should make sure you understand the important skill of representing operators in different coordinate systems, e.g., how to derive expressions for $\nabla f$, $\nabla \cdot \mathbf{v}$, $\nabla \times \mathbf{v}$, $\nabla^2 f$ in cylindrical and spherical geometries.

**Challenge:** Given a stationary state $\psi(x, y, z) = \psi(r, \theta, \phi)$, write down an expression for the probability of finding an electron a distance $r$ from the origin where $r_1 \leq r \leq r_2$.

Considering various operators in cylindrical or spherical coordinate systems is natural since angular momentum classically corresponds to rotation of some object around an axis and it is then natural to choose the rotation axis as the axis of a cylindrical or spherical coordinate system.

7. By solving an operator-eigenvalue problem $L_z f = \lambda f$ and by assuming that the amplitude (not the probability) is single-valued everywhere in space, you should understand how to show that the $z$-component of the angular momentum is discretized and takes on the values $k\hbar$ for $k = 0, \pm 1, \pm 2, \cdots$. The integer denoting the multiple of $\hbar$ is traditionally denoted "$m$" and is called the "magnetic quantum number" An eigenstate of the $z$-component of angular momentum can then be written $|m\rangle$ which means that $L_z|m\rangle = m|m\rangle$.