Lecture 4: Physics 212

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1 Reading

1. Chapter 10 of French and Taylor.
2. Sections 4.3 and 4.4 of Griffiths (angular momentum and spin).

2 Concepts to Understand

After reading Chapter 10 of French and Taylor and related material in the Griffiths book, please make sure you understand the following concepts:

1. You should understand the key idea of Section 7-8, how analyzer loops lead to the idea of a state vector and, further, that a state vector $|\psi\rangle$ can be written as a superposition of two arbitrary orthogonal complete vectors (see Eq. (7-24) on page 301). As an example, an x-polarized photon (photon from an x-polarized beam) passing through a RL-analyzer loop can be written as a superposition of right- and left-circularly polarized light:

$$|x\rangle = |R\rangle\langle R|x\rangle + |L\rangle\langle L|x\rangle \tag{1}$$

$$= \frac{-i}{\sqrt{2}}|R\rangle + \frac{i}{\sqrt{2}}|L\rangle, \tag{2}$$

   where we know the two amplitudes from the phase convention in Table 7-1 on page 299 of French and Taylor This has a rather interesting physical meaning since an R-photon carries a unit $\hbar$ of angular momentum while a L-photon carries a unit $-\hbar$ of angular momentum. Thus an x-linearly polarized photon is a superposition of two kinds of angular momentum and a measurement of angular momentum will yield either $\hbar$ or $-\hbar$ with equal probability. This is completely screwy from a classical view point.

   Puzzle: angular momentum is most definitely a conserved quantity so how is it possible to measure $\pm\hbar$ and not a definite value? If one gets the value $\hbar$, doesn't this violate conservation of angular momentum?

2. French and Taylor did not go quite far enough in their effort to generalize their Eq. (7-19) on page 299. This equation holds not only for any "final state" but also for any initial state $|\psi\rangle$. So Eq. (7-19) could be understood as defining some kind of identity operator I written as follows:

$$I = |y\rangle\langle y| + |x\rangle\langle x|. \tag{3}$$

   The objects on the right sides are linear operators since they convert any vector $|\psi\rangle$ into a new vector while preserving superpositions of vectors. We can call Eq. (3) an "analyzer-loop equation", it says that any state going into an analyzer loop comes out unchanged, but we can think of the state as being separated into orthogonal complete channels during transit.

   More generally, if we have a set of N orthogonal and complete basis vectors $\{|i\rangle\}_i$, we can write down the relation:

$$I = \sum_i |i\rangle\langle i|, \tag{4}$$

   which is a concise mathematical statement of completeness of the basis set and is the generalization of our polarization problem to a analyzer loop with N distinct but complete channels.

3. You should review the concept of a magnetic moment, a three-dimensional vector $\mathbf{m}$ that characterizes the *far-field* magnetic field of a localized but arbitrary *stationary* (time-independent) current distribution. Mathematically, in any simply connected region of space (not an annulus or torus!), one can write a magnetic field $\mathbf{B}(t,x,y,z)$ (or any divergence-free field like the velocity field of an incompressible fluid) in terms of a single-valued vector potential $\mathbf{A}(t,x,y,z)$:

$$\mathbf{B} = \nabla \times A. \tag{5}$$

If one has a static distribution of current density $\mathbf{J}(x,y,z)$ (think of a fixed current I flowing through some arbitrarily complicated folded wire) and if this current density is localized to some region of space, then the vector potential far away from the current density can be written to lowest-order in the distance $\|\mathbf{x}\|$ away from the vicinity of the current density as follows,

$$\mathbf{A}(\mathbf{x}) = \frac{\mathbf{m} \times \mathbf{x}}{\|\mathbf{x}\|^3} + O\left(\frac{1}{\|\mathbf{x}\|^3}\right), \tag{6}$$

where the magnetic moment is defined to be (in CGS units)

$$\mathbf{m} = \frac{1}{2c} \int \mathbf{x} \times \mathbf{J}\, d\mathbf{x}. \tag{7}$$

By taking the curl of Eq. (6), one finds that the magnetic field far away from the current density has, to leading order, a classical dipole structure:

$$\mathbf{B}(\mathbf{x}) = \frac{3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{m}) - \mathbf{m}}{\|\mathbf{x}\|^3} + O\left(\frac{1}{\|\mathbf{x}\|^4}\right), \tag{8}$$

where $\hat{\mathbf{n}} = \mathbf{x}/\|\mathbf{x}\|$ is the unit vector pointing in the direction of the position $\mathbf{x}$.

While undergraduate books (including French and Taylor) tend to suggest that one think of a magnetic moment as a small bar magnet or a small planar current loop, you should appreciate its deeper significance as the leading-order term in a so-called "multipole expansion" of a static current distribution. In particular, the magnetic moment does not usually provide a good description of a magnetic field if one is close to a current distribution. Good discussions can be found in Griffiths E&M book, the classic book on electricity and magnetism by Edward Purcell, and the graduate-level book by J. D. Jackson.

Trivia point: did you know that the earth's magnetic field is not constant over time but flips every 10,000 years or so, with the north and south poles migrating in a complex way until they have exchanged their positions? The earth's magnetic field is generated by a complicated fluid dynamics of a conducting convecting iron core. This dynamics is chaotic (non-periodic and certainly not stationary) and so changes slowly over time.

4. Given some configuration of particles of mass m_i, charge e_i, positions $\mathbf{x}_i$, and velocities $\mathbf{v}_i$, you should understand the relation of the magnetic moment $\mathbf{m}$ of these moving charges (currents generating a magnetic field) to the angular momentum $\mathbf{L}$. The angular momentum vector does not have to be parallel to the magnetic moment (can you think of a trivial physical example of this?) but this becomes the case if all the particles in a system have the same charge-to-mass ratio, $e_i/m_i = e/m =$ constant.

As a clue, one can write the current density $\mathbf{J}(x,y,z)$ of the above discrete group of moving charges in terms of Dirac delta functions as follows:

$$\mathbf{J} = \sum_i e_i \mathbf{v}_i \delta(\mathbf{x} - \mathbf{x}_i) \tag{9}$$

$$= \sum_i \frac{e_i}{m_i} \mathbf{p}_i \delta(\mathbf{x} - \mathbf{x}_i), \tag{10}$$

where $\mathbf{p}_i$ is the momentum of the ith particle. Then Eq. (7) gives:

$$\mathbf{m} = \frac{1}{2c} \sum_i \frac{e_i}{m_i} \mathbf{x}_i \times \mathbf{p}_i \tag{11}$$

$$= \frac{1}{2c} \sum_i \frac{e_i}{m_i} \mathbf{L}_i, \tag{12}$$

where $\mathbf{L}_i$ is the angular momentum (with respect to a particular origin) of the ith particle. From this, we see that the magnetic moment is not necessarily parallel to the total angular momentum

$$\mathbf{L} = \sum_i \mathbf{L}_i. \tag{13}$$

But if all the particles have the same charge-to-mass ratio e/m (e.g., electrons "going around" the nucleus of an atom), the magnetic moment and angular momentum do become parallel, with a known proportionality constant known as the $e/(2mc)$ known as the *gyromagnetic ratio*.

5. You should understand the basic physics of a Stern-Gerlach device, which is a method of sorting particles in a beam according to the z-component of their magnetic moment. If (and this is a big if) the magnetic moment is known to be proportional to the angular momentum, then the Stern-Gerlach device also sorts particles by the z-component of their angular momentum.

   How do we know that $\mathbf{m} \propto \mathbf{L}$ for something like a silver or cesium atom?

6. From Griffiths (also French and Taylor), review your commutation relations for the components of position $\mathbf{x}$, momentum $\mathbf{p} = (\hbar/i)\nabla = (\hbar/i)(\partial_x, \partial_y, \partial_z)$, and angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$. In particular, you should understand (and memorize) the vector commutation relation:

$$\mathbf{L} \times \mathbf{L} = i\hbar\mathbf{L}, \tag{14}$$

   which makes no sense for classical (non-operator) vectors since any vector crossed with itself gives zero.

   You should also recall that the operator corresponding to the magnitude squared of the angular momentum L^2 commutes with all of its components, $[L^2, \mathbf{L}] = 0$ and with the radius $r = \sqrt{x^2 + y^2 + z^2}$, $[L^2, r] = 0$. (Actually one most easily shows that $[L^2, r^2] = 0$, can you see why this would imply $[L^2, r] = 0$?)

7. By solving an operator-eigenvalue problem $L_z f = \lambda f$ and by assuming that the amplitude (not the probability) is single-valued everywhere in space, you should understand how to show that the z-component of the angular momentum is discretized and takes on the values $i\hbar$ for $i = 0, \pm1, \pm2, \cdots$. The integer denoting the multiple of $\hbar$ is traditionally called "m" and so an eigenstate of the z-component of angular momentum can be written $|m\rangle$.

8. Since $[L^2, L_z] = 0$, states can be simultaneously be labeled by their value m of angular momentum in the z direction and by the value of the magnitude squared of the angular momentum. The latter is also discretized and can take on the values $l(l+1)$ for all integers between 0 and l and so one usually classifies a quantum state of angular momentum by two integers, l and m, e.g., $|l, m\rangle$. Griffiths gives a famous derivation of the eigenvalues of L^2 using ladder operators. We will show in a later lecture that the corresponding eigenvectors are the famous spherical harmonics $Y_l^m(\theta, \phi)$, an insight which will be extremely valuable when discussing the quantum mechanics of atoms and nuclei (objects whose Hamiltonians have spherical symmetry about a center)