Tutorial on obtaining Taylor Series Approximations without differentiation

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1 Overview

An important mathematical technique that is used many times in physics, biophysics, and engineering is to obtain low-order (usually quadratic or less) polynomial approximations to functions \( f(x) \) by calculating the first few terms of a Taylor series of the function about some point \( x = x_0 \) of interest. Such approximations can be used to develop or to analyze mathematical models of physical phenomena, and to develop algorithms for analyzing experimental data. A low-order Taylor-series approximation is often the quickest and easiest way to do a calculation that leads to a quantitative scientific insight.

Calculating the Taylor series of a function about some point formally involves calculating and evaluating successive derivatives of the function (see Eq. (6) below) which, by hand, can be tedious and error prone for all but the simplest functions. This tutorial will show you that the first few terms of the Taylor series of many functions that arise in the sciences can be determined easily without any differentiation, by making use of a list of known expansions like this

\[
(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!} x^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!} x^3 + \cdots, \\
e^x = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots, \\
\sin(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots, \\
\cos(x) = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \cdots, \\
\ln(1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 + \cdots,
\]

of Taylor series of elementary functions about the point \( x_0 = 0 \). Here the dots \( \cdots \) stand for the infinitely many other terms in the series that are not being written down, and that can be ignored compared to the written terms if \( x \) is sufficiently small in magnitude (\(|x|\) is sufficiently tiny compared to the radius of convergence of the power series).

Mathematically, using the first few terms of a Taylor series to approximate that function in the vicinity of some point can be considered a simple example of the broader topic of “approximation theory”, which is widely used in mathematics and numerical analysis to approximate a possibly complicated function by a much simpler function like a low-order polynomial, a low-order rational function such as \((a + bx)/(c + dx)\), or perhaps by the first few terms of a Fourier series. It is then usually easier or faster to evaluate, integrate, or differentiate the approximation rather than the original function itself.

Physically, using the first few terms of a Taylor series to approximate a function can be considered one of the simplest examples of what scientists call “perturbation theory”. Although most physical problems are described by mathematical equations that cannot be solved to yield an exact explicit solution, surprisingly many physical problems can be understood as involving small corrections (perturbations) to a problem that does have an explicit concise solution. Perturbation theory represents a collection of techniques for approximating systematically an unknown exact solution as a sum of ever smaller corrections to the known solution of a simpler but physically related problem.

\footnote{You actually need to remember only \textit{two} entries in this list, namely Eqs. (1) and (2). The series for \( \sin \) and \( \cos \) follow from that for \( \exp \) by substituting the imaginary argument \( ix \) into Eq. (2) and by separating out the real and imaginary parts. The series for \( \ln(1 + x) \) follows from Eq. (1) by setting \( \alpha = -1 \) and then by integrating the resulting geometric series \( 1/(1 + x) = 1 - x + x^2 - \cdots \) term by term.}
Interpreting a Taylor series as a perturbation problem is motivated by thinking of the value \( f(x_0) \) of a function \( f(x) \) at a point \( x_0 \) as an exact value and then by asking how the value \( f(x_0) \) is perturbed (changed) if one perturbs the argument \( x_0 \) by some small amount \( \epsilon \), i.e., how is the value \( f(x_0 + \epsilon) \) related to the value \( f(x_0) \) if \( \epsilon \) is a tiny perturbation of \( x_0 \)? The Taylor series (if convergent) gives us a systematic way to calculate the value of \( f(x_0 + \epsilon) \) in terms of successive powers of the small quantity \( \epsilon \) like this:

\[
f(x_0 + \epsilon) = f(x_0) + f'(x_0)\epsilon + \frac{1}{2!}f''(x_0)\epsilon^2 + \frac{1}{3!}f'''(x_0)\epsilon^3 + \cdots \tag{6}
\]

\[
= \sum_{n=0}^{\infty} \left[ \frac{1}{n!}f^{(n)}(x_0) \right] \epsilon^n. \tag{7}
\]

If \( \epsilon \) is sufficiently tiny ("sufficiently" is ambiguous and depends on how much accuracy some scientist is interested in), higher powers of \( \epsilon \) decrease rapidly in magnitude and so higher-order terms in Eq. (6) can often be ignored as negligible, leading to a simple local approximation. For example, for sufficiently small \(|\epsilon|\), all of the terms in Eq. (6) except the first two can be ignored giving a locally linear approximation \( f(x_0 + \epsilon) \approx c_0 + c_1\epsilon \), where \( c_0 = f(x_0) \) and \( c_1 = f'(x_0) \).

A physics example of perturbation theory is the problem of calculating the path of the planet Mercury as it orbits the Sun. If one ignores all of the other planets in the solar system, one ends up with a two-body gravitational problem which is simple enough that Newton’s equations of motion can be solved explicitly and exactly, yielding an elliptical orbit of Mercury about the Sun. Because the Sun is about 1,000 times more massive than all of the other masses in the solar system (including all of the asteroids and comets) and because all of the other masses are far from Mercury compared to the Sun, all of the other masses in the solar system have a tiny gravitational influence on Mercury’s orbit compared to the Sun’s influence. So although one cannot solve Newton’s equations of motion explicitly for the motion of Mercury when all of the other planets are included, one can use a perturbation method to calculate in a systematic way the tiny influence of all of the other planets. The net effect is that the major axis of Mercury’s elliptical orbit very slowly rotates (“precesses”) over time, i.e., the unknown exact solution of the many-body problem is a small and easily calculated correction to the two-body elliptical solution, which remains accurate over long time intervals. (See the URL [http://physics.ucr.edu/~wudka/Physics7/Notes_www/node98.html](http://physics.ucr.edu/~wudka/Physics7/Notes_www/node98.html) for a picture of this precession and for some further details.)

## 2 Efficient calculation of Taylor series

Given these more general comments, let’s start our discussion of how to find low-order Taylor series approximations of a given function about a given point but without calculating any derivatives. From the examples worked out below, you will see that the key steps involve repeatedly adding and multiplying low-order polynomials (usually cubic or less). The addition of two polynomials is simple but the repeated multiplication of polynomials usually leads to bigger and bigger polynomials of higher and higher degree, which is messy. Fortunately, specifically in the context of Taylor series, we can carry out a funny and fast kind of polynomial multiplication in which any power \( x^m \) that appears with a sufficiently high power \( m \) is simply set to zero.

One other trick is needed to give you the full picture, which is to use the generalized binomial theorem Eq. (1) to convert polynomials in a denominator to polynomials in a numerator. We discuss that idea separately a little further below.

Let me show you by an example why calculating Taylor series reduces to the repeated multiplication of low-order polynomials. Consider the function

\[
f(x) = \cos(x) e^x \tag{8}
\]

near the point \( x_0 = 0 \) and let’s approximate this function with a cubic polynomial near \( x = 0 \) by using the first few terms of the Taylor series of \( f \) about \( x = 0 \). Since Eq. (4) and Eq. (2) give polynomial representations of these functions, a natural guess (whose proof I leave to you if you are mathematically inclined) is that if we simply substitute Eqs. (2) and (4) into Eq. (8) and multiply out the two infinite series to get a new infinite series, that resulting infinite series must be the Taylor series representation of \( f \).
But we are interested in the properties of Eq. (8) only for \( x \) near zero, i.e., when the magnitude \(|x|\) of \( x \) is a small number, and you know that raising a small number to a high power results in an even smaller number. (For example if \( x = 10^{-2} \) then \( x^2 = 10^{-4}, x^3 = 10^{-6} \), etc). So instead of substituting the entire infinite series for \( \cos(x) \) and the entire infinite series for \( \exp(x) \) into Eq. (8), it should be ok to use just the first few terms of each series, since we can always make the missing terms sufficiently negligible by taking \(|x|\) sufficiently tiny. Since our goal is to get a cubic approximation (a polynomial in \( x \) of degree three), let’s guess (and find out later that the guess is correct) that we need to retain the terms in the Taylor series up to third-order like this:

\[
\cos(x) e^x \approx \left(1 - \frac{1}{2} x^2 \right) \times \left(1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 \right),
\]

\(9\)

We now proceed to multiply out these two polynomials but set any power of \( x \) higher than \( x^3 \) immediately to zero since we can assume it is too small to be significant, at least if \( x \) is tiny enough. As you should verify, the result is

\[
\cos(x) e^x \approx \left(1 - \frac{1}{2} x^2 \right) \times \left(1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 \right)
\approx \left[1 \right] \times \left(1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 \right) + \left[-\frac{1}{2} x^2 \right] \times \left(1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 \right)
\approx \left(1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 \right) + \left[-\frac{1}{2} x^2 - \frac{1}{2} x^3 + 0 + 0 \right]
\approx 1 + x - \frac{1}{3} x^3,
\]

\(13\)

and this last line is indeed the first three terms of the Taylor series of \( \cos(x) e^x \) about \( x = 0 \), as you can verify tediously and directly via Eq. (6). Note how there was a funny cancellation leading to no quadratic term \( x^2 \) in the answer Eq. (13).

The key step for you to pay attention to is in line Eq. (11), where the \(- \left(1/2 \right) x^2 \) term from the \( \cos(x) \) Taylor series multiplied the \left(1/2 \right) x^2 + \left(1/6 \right) x^3 \) pieces from \( e^x \), giving powers of \( x^4 \) and \( x^5 \) which we simply set to zero in Eq. (12), i.e. we can simply ignore any product that leads to a power greater than 3. Ignoring powers that are higher than the order of approximation (here 3) is what let’s one calculate low-order Taylor series approximations quickly and easily.

The cubic polynomial Eq. (13) should be an arbitrarily good approximation to the function Eq. (8) provided that \(|x|\) is sufficiently small but how small is small? This cannot be answered in general since it involves taking into account the infinitely many terms in the Taylor series that we have ignored. For this particular example, it is easy to just see what is going on by making a plot of Eqs. (8) and (13), say by typing the Mathematica command

\[
\text{Plot[}
\{ \text{Cos[x]} \text{ Exp[x]} , 1 + x - \left( \frac{1}{3} \right) x^3 \} , \{ \text{x, -4, 5 } \} , \text{AxesLabel -> } \{ \text{x , "f, cubic" } \}
\]
\]

\text{to produce this plot:}
The blue curve is Eq. (8) and the gold curve is the cubic Taylor-series approximation Eq. (13). The curves are indistinguishable over the range $-1 < x < 1$ so, for this specific problem, $x$ does not have to be tiny at all for the Taylor series to give a good “local” approximation. (A more careful study shows that the cubic gives a five percent relative error or smaller over the range $-0.6 < x < 0.7$.) Note that the local approximation Eq. (13) does a terrible job in approximating $f$ far from 0, e.g., it completely misses how $f$ starts to increase again around $x = 3.5$.

The last step in calculating low-order Taylor series approximations efficiently is to use Isaac Newton’s powerful generalized binomial theorem

$$(1 + \epsilon)^\alpha = 1 + \frac{\alpha}{1!}\epsilon + \frac{\alpha(\alpha - 1)}{2!}\epsilon^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!}\epsilon^3 + \cdots, \quad \text{for } |\epsilon| < 1,$$

which can be shown to converge for any real-valued exponent $\alpha$ and for any real-valued expression $\epsilon$ whose magnitude is less than one.

There are three practical points to keep in mind when applying Eq. (14). First, for many problems, the first two terms on the right in Eq. (14) suffice for a good approximation, i.e.,

$$(1 + \epsilon)^\alpha \approx 1 + \alpha \epsilon, \quad \text{for } |\epsilon| \ll 1. \quad (15)$$

The quadratic or higher-order powers of $\epsilon$ in Eq. (14) become important mainly when there is some kind of cancellation between two or more terms in some expression that cause the lowest-order terms to cancel each other out. For example, in the expression

$$y = (1 + x)^{1/3} + (1 - x)^{1/3} - 2, \quad (16)$$

where $x$ is small in magnitude compared to one, using just the first two terms of Eq. (14) with $\epsilon = x$ would lead to the wrong conclusion that $y = 0$ and so doesn’t depend on $x$ since, to lowest-order in powers of $x$, we have

$$(1 + x)^{1/3} \approx 1 + \frac{1}{3} x, \quad (1 - x)^{1/3} \approx 1 - \frac{1}{3} x, \quad (17)$$

and so

$$(1 + x)^{1/3} + (1 - x)^{1/3} - 2 \approx \left(1 + \frac{1}{3} x\right) + \left(1 - \frac{1}{3} x\right) - 2 = 0, \quad (18)$$

and so $y$ in Eq. (16) seems to be zero, which cannot be the case since the value $x = 1$ in Eq. (16) leads to a non-zero value of $y$. But if you use the first three terms in Eq. (14) with $\epsilon = x$, you can verify that the terms involving the quadratic power $x^2$ do not cancel and so $y \approx -(2/9)x^2$ to the lowest-power of $x$. More generally, you need to retain enough terms in Taylor series like Eqs. (14)-(15) so that some non-zero...
lowest-order power of $x$ is obtained, which then often provides a useful approximation to the expression of interest near the point of interest.

A second practical point when using Eq. (14) is how to manipulate some expression that does not immediately have the form $(1 + \epsilon)^\alpha$ to have this form. This involves first looking for a sum of several terms that is being raised to some power, and then by seeing if the sum can be partitioned into two parts, with one part being small in magnitude compared to the other part. If so, then it is usually straightforward to manipulate the expression into the form Eq. (14) by factoring out the “not tiny” part of the sum.

For example, if you were given the expression
\[(\ln(z) + \sin(y) + x^2)^{\sqrt{2}}, \tag{19}\]
and you somehow know, say by some scientific context, that the quantities $\sin(y)$ and $x^2$ are always small compared to the value of $\ln(z)$, then indeed the sum $\ln(x) + \sin(y) + x^2$ can be written as the sum of a “large” piece $\ln(z)$ and a “small piece” $\sin(y) + x^2$. Eq. (19) can then be transformed into the form Eq. (14) with $\epsilon = \frac{\text{"small piece of sum"}}{\text{"big piece of sum"}} = \frac{\sin(y) + x^2}{\ln(z)}$ and $\alpha = \sqrt{2}, \tag{20}$
by factoring out the large part $\ln(z)$ like this:
\[(\ln(z) + \sin(y) + x^2)^{\sqrt{2}} = \left[\ln(z) \left(1 + \frac{\sin(y) + x^2}{\ln(z)}\right)\right]^{\sqrt{2}} \tag{21} = \ln(z)^{\sqrt{2}}(1 + \epsilon)^\alpha. \tag{22}\]

We can then apply Eq. (16) to get the lowest-order correction in powers of $\epsilon = (\sin(y) + x^2)/\ln(z)$ to the $\epsilon = 0$ value like this:
\[(\ln(z) + \sin(y) + x^2)^{\sqrt{2}} \approx \ln(z)^{\sqrt{2}} \left(1 + \sqrt{2} \frac{\sin(y) + x^2}{\ln(z)}\right). \tag{23}\]

This example illustrates well the kind of higher-level pattern recognition you have to carry out to manipulate an expression into the form Eq. (14). When one uses series like Eqs. (1)-(5), the quantity $x$ is often not just a single symbol but some expression with multiple pieces.

One does have to be thoughtful when considering which parts of some expression are small or large. For example, in statistical physics one sees expressions like
\[\frac{1}{1 + e^{E/(kT)}}, \tag{24}\]
where $E > 0$ is some positive energy value, $k$ is Boltzmann’s constant, and $T > 0$ is the absolute temperature of a system. One might be tempted to apply Eq. (2) directly with $x = E/(kT)$ but one first has to understand the scientific context of this expression. At high temperatures, which by definition means that $E \ll kT$, the argument $E/(kT) \ll 1$ and it is then indeed ok to use Eq. (2) to start finding a simplified approximation. But at low temperatures such that $E/(kT) \gg 1$, using Eq. (2) would not lead to a concise and so useful approximation since one needs many terms in this series to approximate the exponential of a large argument.

But by realizing that if $e^{E/(kT)}$ is a large number then its reciprocal $e^{-E/(kT)}$ is a small number, we can rewrite Eq. (24) to have the form
\[\frac{e^{-E/(kT)}}{1 + e^{-E/(kT)}} = e^{-E/(kT)} \left(1 + e^{-E/(kT)}\right)^{-1}. \tag{25}\]

We can now apply Eq. (11), not Eq. (2), with $\alpha = -1$ and $x = e^{-E/(kT)}$ to obtain a low-order Taylor series approximation in powers of the small quantity $e^{-E/(kT)}$. So the same mathematical expression can require different approximation strategies depending on the scientific context.
The third practical point, the most important one for most calculations, is to use the generalized binomial theorem to convert expansions in a denominator to expansions in a numerator, which greatly simplifies obtaining the low-order Taylor series approximation of some expression. You achieve this by writing a “power of a sum” that appears in a denominator as a power of a sum in a numerator before applying Eq 14.

To illustrate this idea, let’s say that we want to obtain a low-order polynomial approximation to the mathematical expression
\[
\left(\frac{2 + y}{3 - z^2}\right)^{1/3},
\]
where we are told that \(y\) and \(z^2\) are expressions whose magnitudes are small compared to 2 and 3 respectively. If we set \(y = 0\) and \(z^2 = 0\) in Eq. (26), we get the “zeroth-order” or unperturbed value \(2^{1/3}/3\), and we then want to calculate what are the lowest-order corrections to this value when \(y\) and \(z^2\) are small but finite. We start the process of applying Eq. (14) by factoring out the “large” quantities of each sum like this:
\[
\left(\frac{2 + y}{3 - z^2}\right)^{1/3} = \left(\frac{2}{3}\right)^{1/3} \left(\frac{1 + y/2}{1 - z^2/3}\right)^{1/3}.
\]
(27)

where the small quantities will be \(y/2\) and \(-z^2/3\). We could then write Eq. (27) as the ratio of two sums raised to powers like this
\[
\left(\frac{2}{3}\right)^{1/3} \frac{(1 + y/2)^{1/3}}{(1 - z^2/3)^{1/3}},
\]
(28)
and then apply Eq. (14) to the numerator and separately to the denominator, but that would lead to a complicated denominator that is difficult to work with:
\[
\left(\frac{2}{3}\right)^{1/3} \frac{1 + (1/3)(y/2) + \cdots}{1 - (1/3)(z^2/3) + \cdots}.
\]
(29)

It is better first to convert the power in the denominator of Eq. (27) to a power in the numerator by simply reversing the sign of the exponent like this
\[
\left(\frac{2 + y}{3 - z^2}\right)^{1/3} = \left(\frac{2}{3}\right)^{1/3} \frac{(1 + y/2)^{1/3}}{(1 - z^2/3)^{1/3}}
\]
(30)
\[
= \left(\frac{2}{3}\right)^{1/3} \left(1 + \frac{y}{2}\right)^{1/3} \left(1 - \frac{z^2}{3}\right)^{-1/3}
\]
(31)
\[
\approx \left(\frac{2}{3}\right)^{1/3} \left[1 + \left(\frac{1}{3}\right)\frac{y}{2}\right] \left[1 + \left(-\frac{1}{3}\right)\left(-\frac{z^2}{3}\right)\right],
\]
(32)
\[
\approx \left(\frac{2}{3}\right)^{1/3} \left[1 + \frac{1}{6}y + \frac{1}{9}z^2 + \frac{1}{54}yz^2\right],
\]
(33)
\[
\approx \left(\frac{2}{3}\right)^{1/3} \left[1 + \frac{1}{6}y + \frac{1}{9}z^2\right].
\]
(34)

In the last line, I dropped the term involving \(yz^2\) since, as the product of two tiny terms, it is smaller than \(y\) or \(z^2\) and so can be ignored if \(y\) and \(z^2\) are small enough. So Eq. (34) is the lowest-order polynomial approximation to Eq. (26).

3 Some problems to test and improve your understanding

You now know all the key ideas for efficiently obtaining Taylor series approximations of general functions. Try solving the following problems to check that you understand the key ideas and to gain experience.

\(^2\)For functions that diverge at some point \(x_0\), approximating the function by a rational function is more useful than by a Taylor series so one would stop with Eq. (28).
1. Without calculating any derivatives, show that the third-order Taylor-series approximation of the function \( \exp(\sin(x)) \) about \( x = 0 \) is given by

\[
\exp(\sin(x)) = e^{\sin(x)} \approx 1 + x + \frac{1}{2} x^2.
\]

Note: the next power after \( x^2 \) in the Taylor series Eq. (35) is \( x^4 \) so this really is the correct expression up to cubic order.

2. Without using a calculator (completely by hand), use Eq. (14) to show that

\[
\sqrt{\frac{(1.01)^{7/5} + (0.99)^{7/5}}{2}} \approx 1.000014000,
\]

to ten significant digits. Make sure you show clearly how you got your answer without using a calculator, and make sure to explain why, for the right side of Eq. (36), the right most three zero digits (the ones after the digits 14) are significant.

3. Show that the hyperbolic tangent function \( \tanh(x) \) is accurately approximated by the following cubic polynomial for sufficiently small \( x \):

\[
\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \approx x - \frac{1}{3} x^3.
\]

Note that you could have deduced that the lowest-order Taylor series had to have the form \( c_1 x + c_3 x^3 \) without any calculation because \( \tanh(-x) = -\tanh(x) \) is an odd function so that all even powers in the Taylor series must vanish.

4. The Langevin function \( L(x) \) is defined by

\[
L(x) = \coth(x) - \frac{1}{x},
\]

where the hyperbolic cotangent function \( \coth(x) = 1/\tanh(x) = (e^x + e^{-x}) / (e^x - e^{-x}) \). This function appears in statistical physics and biophysics, e.g. when studying how a paramagnetic material loses its magnetism as the temperature increases, or how the length \( L \) of a DNA molecule in thermodynamic equilibrium depends on the force \( f \) that is stretching it (say by optical tweezers). Show that for \( |x| \) sufficient small,

\[
L(x) \approx \frac{1}{3} x - \frac{1}{45} x^3,
\]

and use Mathematica to determine empirically over what range of \( x \) the approximation Eq. (39) gives at least two significant digits correctly.

Note: because there are cancellations in the expression \( e^x - e^{-x} \) in the denominator of \( \coth(x) \), one has to go to higher-order in the denominator than in the numerator to get Eq. (39).

5. From your introductory physics course, you might remember that a current flowing through an idealized infinitely long solenoid has the remarkable property that the magnetic field inside the solenoid is everywhere uniform in direction and magnitude. In this problem, you use a Taylor series approximation to get a sense of how much the magnetic field near the center of a long but finite solenoid differs from that of the infinite solenoid as the solenoid becomes longer and longer. (This is a practical question since scientists use solenoids to create approximately uniform magnetic fields near their center.)

Consider a finite cylindrical solenoid of length \( L \) and of radius \( a \) that is carrying on its surface a uniform azimuthal current density of \( K \) amperes per meter. If the axis of the solenoid is the \( z \) axis with the center of the solenoid located at the origin \( x = y = z = 0 \) (so the ends of the solenoid...
lie at \( z = \pm L/2 \), one can show that the \( z \)-component of the magnetic field \( B_z(z) \) generated by the solenoid at coordinate \( z \) on the axis is given by the following analytical expression:

\[
B_z = \frac{\mu_0 K}{2} \left( \frac{z + L/2}{\sqrt{(z + L/2)^2 + a^2}} - \frac{z - L/2}{\sqrt{(z - L/2)^2 + a^2}} \right),
\]

(40)

where \( \mu_0 \) is the vacuum permittivity.

(a) Show that, for fixed values of \( z \) and \( a \),

\[
\lim_{L \to \infty} B_z = \mu_0 K.
\]

(41)

The value \( \mu_0 K \) is indeed the magnitude of the magnetic field at any point inside an infinitely long solenoid.

(b) Now assume that the solenoid length is finite but much larger than the value of \( z \) (so we are considering points on the \( z \)-axis that are close to the center of the solenoid and far from the ends of the solenoid), and also much larger than the solenoid radius \( a \). Use Eq. (14) and Eq. (40) to calculate the lowest-order correction to the infinitely long solenoid value \( \mu_0 K \). Your answer will have the form:

\[
B_z(L) = \mu_0 K + \frac{c}{L^d},
\]

(42)

where \( c \) is an expression that will contain the symbols \( a \) and \( z \) and where \( d \) will be some exponent that emerges from your calculation. This is a neat insight, it tells you how rapidly the magnetic field near the center of a finite solenoid approaches that of an infinite solenoid as the length \( L \) becomes large (but not infinite).

(c) Now consider the case where the length \( L \) of the solenoid is small compared to the solenoid’s radius \( a \), \( L \ll a \), so that the solenoid approximates a ring of current of radius \( a \), for which the analytical solution for \( B_z \) at \( z = 0 \) is known to be:

\[
B_z(0) = \frac{\mu_0 I}{2R}.
\]

(43)

i. Show that for Eq. (40),

\[
\lim_{L \to 0} B_z(z = 0) = \frac{\mu_0 I}{2a}.
\]

(44)

This is the correct mathematical expression for the magnetic field at the center of a wire ring of radius \( a \) carrying a current \( I \).

ii. Use a Taylor series expansion of Eq. (40) for \( L \ll a \) to calculate the leading-order correction to the \( L = 0 \) limit for \( B_z(z = 0) \).

4 Summary

To summarize tutorial, here is a list of key steps to consider when calculating a low-order polynomial approximation of some function \( f(x) \) about some point \( x_0 \), when the function is a composite of elementary functions whose Taylor series are already known.

1. Before doing any calculation:

   (a) Identify the small quantity or small quantities in the calculation. This often requires knowing the scientific context of the problem. One will obtain an expansion in successive powers of the small quantities.
(b) Evaluate your expression when the small quantity is set to zero so that you know right away what is the basic value to which you will be calculating small corrections. That is, make sure you know what is the zeroth-order approximation to your expression.

(c) See if the function has odd or even symmetry about $x_0$, which can eliminate half of the terms in the series. If the function diverges to infinity at the point of interest, calculate a Taylor series approximation of the reciprocal $1/f$ of the function about the point.

2. See if there is one or more sums of similar expressions that have opposite signs, like Eq. (16) above. Such opposite signs often lead to a final Taylor series with many terms missing because of cancellations.

3. Start replacing elementary functions with the first few terms of their Taylor series for the “deepest” functions in the sense of function composition. That is, if you can write the function in the form $f(g(h(x)))$, replace $h(x)$ with the first few terms of its Taylor series, then simplify $g(h(x))$, and so on.

4. Use the smallest number of terms in each Taylor series, say two (linear order) and only go to higher order if one finds that the lowest-order correction is zero.

5. To avoid working with rational functions, use the generalized binomial theorem to move all powers in denominators to powers in the numerators by reversing the sign of the power.

6. If you have access to a symbolic manipulation program like Mathematica or Maple, use its Taylor series command to verify your calculation and also use its plotting capabilities to determine the range over which the Taylor series approximation is suitably accurate.