LHC lecture this Friday, 5pm

Points from last lecture

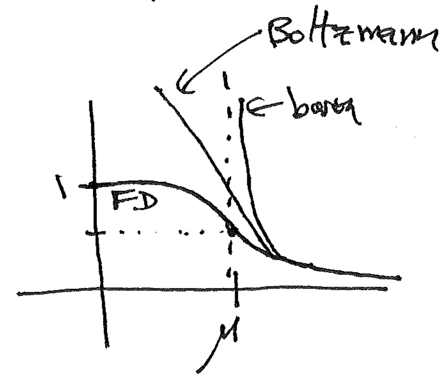
Criterion for when quantum effects of ideal gas are substantial so can not use $Z_N \simeq \frac{1}{N!} Z_1^N$:

$$Z_1 \lesssim N \quad \text{or} \quad \frac{V/N}{V_Q} \lesssim 1$$

equivalent to thermal de Broglie wavelength $\sim (V/N)^{1/3}$ = average interpart spacing

$$\langle E \rangle = \sum_m E_m \langle n_m \rangle \qquad \langle n_m \rangle = \text{occupancy of } m^{th} \text{ level}$$

for fermions: $\bar{n}_{FD}(\epsilon) = \dfrac{1}{e^{\beta(\epsilon-\mu)}+1}$



bosons $\bar{n}_{BE}(\epsilon) = \dfrac{1}{e^{\beta(\epsilon-\mu)}-1}$

classical $\bar{n}_{Boltz}(\epsilon) = N \dfrac{e^{-\beta\epsilon}}{Z_1}$

From $\langle E \rangle$, get $C = \dfrac{d\langle E \rangle}{dT}$, $P = -\dfrac{\partial \langle E \rangle}{\partial V}$, $S = \displaystyle\int_0^T \frac{C(T')}{T'} dT'$

Section 7.3: Application of Fermi-Dirac statistics to ideal spin-1/2 gas of identical fermions, e.g. electrons in metal, neutrons in neutron star, neutrinos, He^3 atoms

For main example, consider electrons in a metal, make two extreme assumptions:

(1) <u>electrons are non-interacting</u>, even though average spacing is small and electrons repel each other. Landau liquid theory, idea of quasiparticle excitations

(2) <u>electrons act as free particles</u>, can ignore interaction of electrons with periodic array of positive nuclei, as if positive charge smeared out into uniform positive jelly ("jellium"). This is actually the less well satisfied approximation, including periodic pos. nuclei leads to band theory. See Ashcroft + Mermin "Solid State Physics" for details. (graduate level book).

Assumptions hold well for many but not all metals, good for alkali metals like Ag, K, Na, bad for transition metals like Fe.

For silver Ag, $\frac{V}{N} \approx (0.2\text{nm})^3$ and $\frac{V_Q}{V/N} \approx 10^5$

so $Z_1 \ll N$, many particles in same energy levels so can't use $Z_N \approx \frac{1}{N!} Z_1^N$

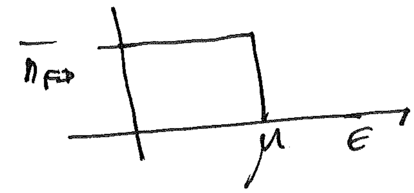
Easiest to proceed by studying zero temp limit $T=0$ first, math is much easier and result turns out to be relevant for nearly all temperatures on Earth.

We have:

$$\langle E\rangle = \sum_s E_s \langle n_s\rangle = 2\sum_{n \,\in\, \text{quantum \#s}} E(n)\,\bar{n}_{FD}\left[E(n)\right]$$

from spin-1/2 assumption, each ~~elec~~ state is degenerate for ↑, ↓ spin

For $T=0$, $\bar{n}_{FD}(E(n))$ has simple form



$$\bar{n}_{FD} = 1 \quad E \leq \mu$$
$$= 0 \quad E > \mu$$

all energy levels are filled up to μ. Highest occupied energy level is called the Fermi energy E_F so

$$\boxed{E_F = \mu \text{ at } T=0}$$

We then have:

$$\langle E\rangle = 2\sum_n E(n)\cdot \bar{n}_{FD}(E(n)) = 2\sum_{\substack{n \\ E_n \leq \mu}} E_n$$

Vocabulary: electron gas for which all lowest energy states are occupied called a "degenerate gas", properties require QM to understand.

For non-interacting free particles in cubic box of length L, walls are impenetrable (so wave f^n of particles is zero outside box), QM tells us that

$$E_s = E(n_x, n_y, n_z) = \frac{h^2}{8mL^2}(n_x^2+n_y^2+n_z^2) \qquad n_x, n_y, n_z \geq 1$$

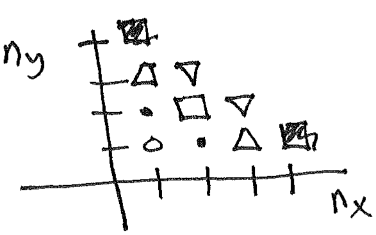
$$= \frac{h^2}{8mL^2} n^2 \qquad n^2 = n_x^2+n_y^2+n_z^2$$

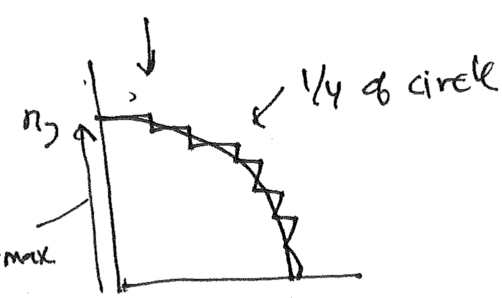
More advanced discussions show that, as long as system is big compared to $\lambda_D = \frac{h}{\sqrt{2\pi m k T}}$, shape of box, type of walls doesn't matter.

If we have N particles, start filling energy levels, starting with lowest, 2 electrons for each energy value since, in absence of external magnetic field, $\uparrow$ and $\downarrow$ have same energy. Claim: for N large, electrons fill up 1/8 of sphere in space (n_x, n_y, n_z), up to some maximum radius n_{max}.

Example for 2D case, $E(n_x, n_y) = \frac{h^2}{8mL^2}(n_x^2+n_y^2)$ $\qquad \epsilon = \frac{h^2}{8mL^2}$

s	n_x	n_y	d_s	$E(n_x,n_y)/\epsilon$
1	1	1	1	2
2	1	2	2	5
3	2	1		
4	2	2	1	8
5	1	3	2	10
6	3	1		
7	2	3	2	13
8	3	2		
9	1	4	2	17
10	4	1		





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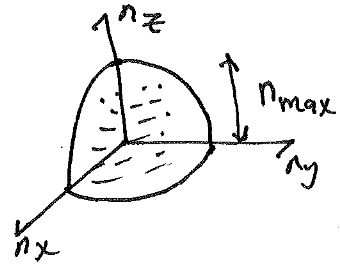
For 3d gas of electrons



For $N \gg 1$, particles occupy 1/8 of sphere of radius n_{max}, so

$$N = \underset{\uparrow\ \text{spin-}1/2}{2} \cdot \underset{\text{positive octant}}{\frac{1}{8}} \cdot \frac{4\pi}{3} n_{max}^3 = \frac{\pi}{3} n_{max}^3$$

so
$$\boxed{n_{max} = \left(\frac{3}{\pi} N\right)^{1/3}}$$

Fermi energy, highest possible electron energy, therefore given by

$$\epsilon_F = \text{Fermi energy} = \frac{h^2}{8mL^2} n_{max}^2 = \frac{h^2}{8mL^2}\left(\frac{3}{\pi}N\right)^{2/3}.$$

or
$$\boxed{\epsilon_F = \frac{h^2}{8m}\left(\frac{3}{\pi}\frac{N}{V}\right)^{2/3}}$$

ϵ_F intensive variable since $\epsilon_F = \epsilon_F(N/V)$ doesn't depend on size or shape of metal volume.

$2/3 \to 1/3$ for relativistic particle $\epsilon \to pc$

Note that, as number density $\frac{N}{V} \uparrow$, $\epsilon_F \uparrow$, electrons move faster in dense electron gas.

For $m = m_e \approx 10^{-30}$ kg, $\frac{N}{V} \approx \frac{10^{23}}{cm^3} \approx 10^{29} m^{-3}$, $h \approx 7 \cdot 10^{-34}$ Js

$$\epsilon_F \cong 7eV \Rightarrow T_F = \frac{\epsilon_F}{k} = \text{Fermi temperature} \approx 80{,}000 K!$$

For room temperature, $T = 300 \ll 80{,}000 K$ so room temperature effectively $T = 0$ for electron gas.

$$\epsilon_F = \frac{p^2}{2m} = \frac{1}{2}mv^2 \Rightarrow \frac{v}{c} \approx 10^{-2},$$ relativistic speeds of highest energy electrons

Zero temperature gas: P, U, S

Now calculate $\langle E \rangle$ at $T=0$ for degenerate electron gas

$$\mathcal{U} = \sum_s E_s \, \bar{n}_{FD}(E_s)$$

$$= 2 \sum_{n_x} \sum_{n_y} \sum_{n_z} \epsilon(n_x, n_y, n_z) \, \bar{n}_{FD}[\epsilon(n_x, n_y, n_z)] \quad \text{general case}$$

$$\cong 2 \int_0^\infty dn_x \int_0^\infty dn_y \int_0^\infty dn_z \, \epsilon(n) \quad \text{since } \bar{n}_{FD} = 1 \quad \epsilon \le \mu; \quad = 0 \quad \epsilon > \mu$$

$\epsilon(n) \le \mu$

$$\simeq 2 \int_0^{n_{max}} dn \int_0^{\pi/2} d\varphi \int_0^{\pi/2} d\theta \; \frac{h^2 n^2}{8mL^2} \times \underbrace{n^2 \sin\theta}_{\text{Jacobian or phase space differential}}$$

$$\cong 2 \cdot \frac{h^2}{8mL^2} \int_0^{\pi/2} d\varphi \int_0^{\pi/2} d\theta \sin\theta \int_0^{n_{max}} n^4 \, dn$$

$$\Rightarrow \quad \mathcal{U} = \langle \epsilon \rangle = \frac{\pi h^2 n_{max}^5}{40 m L^2} = N \cdot \frac{3}{5} \epsilon_F \qquad \epsilon_F = \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3}$$

so average electron energy is: $\frac{\mathcal{U}}{N} = \frac{3}{5} \epsilon_F$

at $T=0$, $S=0$, so $F = \mathcal{U} - TS = \mathcal{U}$

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = -\left(\frac{\partial \mathcal{U}}{\partial V}\right)_{T,N} = \frac{2}{5} \cdot \frac{N}{V} \epsilon_F = \frac{2}{3} \frac{\mathcal{U}}{V}$$

$$\boxed{P = \frac{3}{20} \frac{h^2}{m} \left(\frac{N}{V}\right)^{5/3}}$$

<u>degeneracy pressure</u> - arises from fermion statistics, <u>not</u> from electrical repulsion of electrons

higher the density $\frac{N}{V}$, the greater the pressure, important for white dwarf stars, other stars.

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as $\frac{N}{V}$ increases, ϵ_F increases, $\frac{v}{c}$ increase and particles at top of energy levels, near ϵ_F, need to be treated relativistically, with energy $\epsilon = \sqrt{p^2c^2 + m^2c^4} \simeq pc$ if $pc \gg mc^2$, can ignore rest mass compared to particle energy.

Then $\epsilon(n_x, n_y, n_z) = c\sqrt{p_x^2+p_y^2+p_z^2}$ $\qquad p_i = \frac{hn_i}{2L} \quad i=x,y,z$

$$= \frac{hc}{2L}\sqrt{n_x^2+n_y^2+n_z^2} = \frac{hcn}{2L}$$

You show in Problem 7.22 of Schroeder,

$$\langle E \rangle = \frac{3}{4} N \epsilon_F \quad (\text{vs. } N \cdot \tfrac{3}{5}\epsilon_F \text{ for nonrelativistic case})$$

$$P \propto \left(\frac{N}{V}\right)^{4/3} \quad \text{vs} \propto \left(\frac{N}{V}\right)^{5/3} \text{ for nonrelativistic case}$$

innocent change of $5/3 \to \frac{4}{3}$ has profound implications, mean that white dwarf star above a certain ~~radius~~ mass $\simeq 1.4\, M_{sun}$ can't hold back gravity be degeneracy pressure so white dwarf collapses, leading to heating of core of He/C nuclei, huge explosion that can be seen billion light years away.

Some #s: Cu $\rho = 8.9 \frac{g}{cm^3}$, $m_{Cu} = \frac{63.5 g}{mole}$ from periodic table

$$\frac{V}{N} = \frac{\text{mass}}{\text{density}} = \frac{63.5 g}{8.9 \, g/cm^3} \simeq 7 \cdot 10^{-6} m^3$$

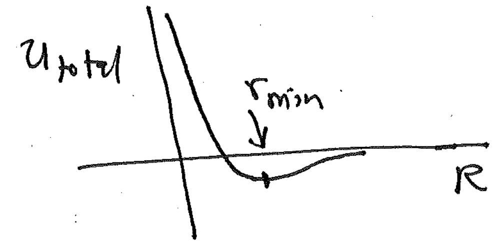
$\frac{N}{V} \simeq 8 \cdot 10^{28} m^{-3}$, $\epsilon_F \simeq 7 eV$, $T_F \simeq 82{,}000 K$, $P \simeq 10^5 atm$

radius of white dwarf determined by attraction

$\mathcal{U} = -\frac{3}{5}\frac{GM^2}{R}$ (uniform density, mass M, radius R) and

degeneracy pressure repulsion,

$$\mathcal{U}_{\cancel{kin}\,tot} = -\frac{3}{5}\frac{GM^2}{R} + C \cdot \frac{M^{5/3}}{R^2}$$



for relativistic electrons, get instead

$$\mathcal{U}_{total} = -\frac{3}{5}\frac{GM^2}{R} + \tilde{C}\,\frac{M^{4/3}}{R} = \left(-\frac{3}{5}GM^2 + \tilde{C}\,M^{4/3}\right)\cdot\frac{1}{R}$$

so can't balance attraction and repulsion, star collapses or expands based on sign of $-\frac{3}{5}GM^2 + \tilde{C}M^{4/3}$, but this becomes negative for big enough M since M^2 grows faster than $M^{4/3}$, hence kaboom if M is too big.

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Heat capacity of electron gas

If electrons in metals were classical free particles, they would contribute amount

$$C_V = N \cdot F \cdot \frac{kT}{2} = \frac{3}{2} Nk$$

to heat capacity of metal, and contribution is indep. of temp. T.

only see vibrational contr. 3Nk

Expts show strong disagreement, C_V from electrons is tiny compared to $\frac{3}{2}Nk$ and is proportional to temp. T.

Won't discuss details (see pages ~~278-279~~ 277-278 of Schroeder), but give qualitative idea: only electrons within energy distance $kT \ll E_F$ have nearby empty states they can transition to as T increases, gives

$$C_V \sim C_V^{class} \times \frac{kT}{\epsilon_F} \propto T$$

correct prediction

More careful calculation gives

$$C_V^{elect} = \frac{\pi^2}{2}\left(\frac{kT}{\epsilon_F}\right) Nk$$

free electron gas, jellium approx

Some #s: for Cu,

$$C_V^{elect} \approx 0.2\,Nk$$

$$C_V^{vibr} \approx 3Nk$$

at $\approx T = 300K$

$$\frac{C_V^{elect}}{C_V^{vibr}} \lesssim 1\% \text{ at room temp}$$