Outline of lecture 2/15/2011

1. Finish discussion of Einstein solid, why is multiplicity \( S_{\text{total}}(\theta_A) = S_A(\theta_A) S_B(\theta_B) \), such a fantastically narrow peak for \( N_{A_{\text{low}} N_{B_{\text{low}}}} \)
   - sub theme: Functions raised to high powers look like Gaussian near global maximum

2. Heuristic derivation of multiplicity of ideal gas, \( S[N,Y,V] \), for atoms
   - discuss Gamma \( \Gamma(x) \)
   - surface area of d-dimensional hypersphere

3. Implications of \( S[N,Y,V] \):
   - Sackur-Tetrode equation for entropy \( S[N,Y,V] = k \ln S \)
   - \( S \) very sharply peaked \( f^n \) of \( V_A, V_B, N_A \) for two subsystems of gases \( A, B \)
   - entropy of mixing
Why Are Some Dimensionless Physical Ratios So Big or So Small?

Fina Juning By Deity, Lottery Ticket?

Some basic constants of nature look big or small but this is misleading, result of choice of unit, like m, s, kg

\[ h \approx 10^{-33} \text{ J.s} \quad G \approx 10^{-10} \frac{\text{N.m}^2}{\text{kg}^2} \]

\[ c \approx 10^8 \frac{\text{m}}{\text{s}} \quad e \approx 10^{-19} \text{C} \]

\[ M_e \approx 10^{-30} \text{kg} \]

But what about dimensionless ratios of similar quantities this gives pure number that has same value no matter what units are used.

Example: consider hydrogen atom from classical view point, what is ratio of electrical force to gravitational force?

\[ F_{\text{electrical}} = \frac{k e^2}{r^2} \quad k \approx 10^{10} \frac{\text{Nm}^2}{\text{C}} \]

\[ F_{\text{gravitational}} = \frac{G M_e m_p}{r^2} \quad G \approx 10^{-10} \]

\[ \frac{F_{\text{elec}}}{F_{\text{grav}}} = \frac{k e^2}{G M_e m_p} \approx 10^{37} \quad \text{a large # no matter what units are used} \]
why would any dimensionless ratio be so large (or its reciprocal so small)? Why not of order 1 or 10 or 100? 

Dirac had insight when observed

\[
\frac{\text{diameter of universe}}{\text{diameter of proton}} \sim \frac{1 \times 10^{10} \text{ ly} \times 10^{16} \text{ m}}{10^{-15} \text{ m}} \sim 10^{41}
\]

maybe two such large numbers must be related, one explains the other.

But universe is expanding (Hubble 1922), so if related conclude that

\[
\frac{k e^2}{G M e m_p}
\]

must vary with time, i.e. some fundamental constants are not constant!

Can test by using big telescopes to look far out in space and so look back in time, 10^10 years in past, no evidence for time dependence. So why the big ???
Fine Tuning

One reason why Dirac was wrong is that he didn't consider smallest possible physical length, the Planck length

\[ l_p = \left( \frac{\hbar G}{c^3} \right)^{1/2} \approx 10^{-35} \text{ m} \]

\[ t_p = \frac{l_p}{c} = 10^{-43} \text{ s} \]

\[ m_p = \sqrt{\frac{h c}{G}} \approx 10^{-8} \text{ kg} \]

\[ \text{ratio} \frac{\text{size of universe}}{\text{Planck length}} \approx \frac{10^{10} \text{ ly} \times 10^{16} \text{ m}}{10^{35}} \approx 10^{61} \]

# of Planck volume in universe \( \approx (10^{61})^3 \approx 10^{180} \), much much smaller than Graham's # so physically impossible to write out that number

Strange fact: as big as Feke/Fyew is, small change in magnitude, say \( 10^{35} \) or \( 10^{34} \), would lead to different kind of universe.

stars would have lifetime \( < 10^6 \) y \( \Rightarrow \) no evolution

stars fail to undergo fusion \( \Rightarrow \) no periodic table \( \Rightarrow \) no life

Fine tuning of dimensionless quantities a mystery + nuisance

read "Just Six Minutes" by Martin Rees
Voyager II plague

show on projector

≈ 1972

how to identify length scale, time scale, energy scale with alien civilization?

Linear mathematical language: possible to communicate across worlds?
Key insight from previous lecture

\[ S(N, \theta) = \binom{N+\theta}{\theta} \]

\( N \) quantum oscillators with energy \( \theta \)

\[ S_{\text{total}}(\theta) \equiv S_A(\theta) S_B(\theta) \]

two interacting subsystem

\[ \theta = \theta_A + \theta_B \]

some macrostates much more likely than others if all accessible microstates are equally likely

\[ S(N, \theta) \approx \left( \frac{e \theta}{N} \right)^N \quad \text{for} \quad \theta \gg N \gg 1 \]

high temperature limit

make sure you can carry out this derivation using Stirling's formula \( \ln(n!) \sim n \ln(n-n) \) and Taylor series approx \( \ln(a+\varepsilon) \approx \ln(a) + \frac{\varepsilon}{a} \) if \( |\varepsilon| < 1 \)
Application of High-Temp Einstein

Multiplicity: Show \( C_V \) satisfies

Equation: \( C_V = NkT \quad \text{F} = 2 \)

\[
S(N, q) = \left( \frac{q}{N} \right)^N \quad \Rightarrow \quad S = kN \ln S = Nk \ln \left[ \frac{e^q}{N} \right]
\]

But \( U = (kT)q = e^q \) if we define \( e = \hbar \nu \) = energy level spacing

So

\[
S = S(N, U) = Nk \ln \left[ \frac{e}{N e^U} \right] = Nk \ln \left[ \frac{e}{N} \right] + Nk \ln U
\]

constant

\[= a + b \ln U\]

This is the recent quiz problem, we can calculate \( C_V \) by first deducing \( T = T(U) \), then solving for \( U = U(T) \) then differentiating to give \( C_V = \frac{dT}{dU} \)

\[
\frac{1}{T} = \left( \frac{\partial S}{\partial U} \right)_N = \frac{Nk}{U} \quad \Rightarrow \quad U = NkT
\]

\( \Rightarrow \quad C_V = Nk = N \cdot \Phi \left( \frac{e}{2} \right) \) with \( T = 2 \)

so high-temperature regime of Einstein solid agrees with equipartition.

In homework, you work out entire form \( C_V(T) \), study low and high-\( T \) approximations.
Multiplicity of two coupled macroscopic Einstein solids has extremely narrow peak:

\[ \text{width} \propto \frac{1}{\sqrt{N}} \approx 10^{-12} \text{ for } N \approx 10^2 \]

To make discussion easier, assume equal number of identical non-interacting quantum harmonic oscillators in each subsystem

\[ N_A = N_B = N \gg 1 \]

Also assume high-temperature regime: \( q \gg N \gg 1 \), a lot of energy in not so many oscillators

What is probability of observing macrostate with energy \( q_A \) in \( A \), energy \( q_B = q - q_A \) in \( B \)?

\[ S_{\text{total}} (q_A) \approx S_A (q_A) S_B (q_B) \]

\[ \approx \left( \frac{e^{q_A}}{N} \right)^N \left( \frac{e^{q_B}}{N} \right)^N \]

\[ S_{\text{total}} = \left( \frac{e}{N} \right)^{2N} \left[ \frac{q_A (q - q_A)}{N} \right]^N \]

Function \( q_A (q - q_A) \) raised to high power has single maximum of \( \frac{q_A^2}{4} \) at \( q_{\text{max}} = \frac{q_A}{2} \)
Theorem: \( f(x)^N \) raised to high power

\( N \) looks like Gaussian centered on the global maximum of \( f(x) \)

\( \text{not too hard to see that peak shrinks in width as } N \text{ increases} \)

\( f(x)^N = f_{\text{max}} \left[ \frac{f(x)}{f_{\text{max}}} \right]^N \)

but high powers of \( #<1 \) less than one become quite small, only

numbers near peak for which \( f(x)/f_{\text{max}} \approx 1 \) survive since

\( 1^N = 1 \) for all \( N \)

Use two standard tricks to analyze this situation

- take logarithm of \( f(x)^N \) so we deal with large
  numbers rather than very large numbers

- use Taylor series to low order to get useful
  approximation about location of maximum
Let's see how it goes

\[ g(x) = \left[ F(x) \right]^N \]
define \( g \) for convenience
\( \text{assume } f(x) \) to everywhere

\[
\ln g(x) = N \ln F(x) \approx 0 \text{ since expanding around maximum} \\
\approx N \ln \left[ F(x_{\text{max}}) + f'(x_{\text{max}})(x - x_{\text{max}}) \right. \\
+ \frac{1}{2} f''(x_{\text{max}}) (x - x_{\text{max}})^2 \\
\left. + \text{ higher order terms} \right]\]

\[
\ln g(x) \approx N \ln F(x_{\text{max}}) + N f''(x_{\text{max}}) \frac{(x - x_{\text{max}})^2}{2 f(x_{\text{max}})} 
\]

where I used our favorite trick with logs:

\[
\log(e^x + e^y) \approx \log e^y + \frac{e^x}{y} \text{ if } |x/y| \ll 1
\]

Now exponentiate both sides of (x) to undo logarithm. Get

\[
g(x) = \left[ F(x) \right]^N \approx F_{\text{max}}^N \exp \left[ N \frac{f''(x_{\text{max}})}{2 f(x_{\text{max}})} (x - x_{\text{max}})^2 \right]
\]

remember that \( f''(x_{\text{max}}) < 0 \) near a maximum since \( f(x) \)
is changing from positive to negative about peak

\[
f_{\text{max}}^N = \frac{f(x_{\text{max}} + \epsilon) - f(x_{\text{max}} - \epsilon)}{2 \epsilon} > 0
\]
Apply "Theorem" to coupled Einstein solid

\[ f(\theta_A) = \theta_A(\theta_A - \theta_B) = \theta_A^2 - \theta_A \theta_B \] \[ \text{constant} \]

\[ f'(\theta_A) = -2\theta_A + \theta = 0 \quad \text{at} \quad \theta_A = \theta/2 = \theta_{\text{max}} \]

\[ f(\theta_{\text{max}}) = \frac{\theta^2}{4} \]

\[ f''(\theta_A) = -2 \quad \text{independent of} \quad \theta_A \]

So:

\[ \left[ \theta_A(\theta_A - \theta_B) \right]^N \sim \left( \frac{\theta/4}{N} \right)^N \exp \left[ -\frac{N}{2} \cdot \frac{2}{\left( \frac{\theta/4}{\theta_{\text{max}}} \right)^2} \right] \]

\[ \sim \left( \frac{\theta/4}{N} \right)^N \exp \left[ -N \left( \frac{\theta_A - \theta_{\text{max}}}{\theta_{\text{max}}} \right)^2 \right] \]

So:

\[ \Sigma_{\text{total}} = \Sigma_A(\theta_A) \Sigma_B(\theta_B) = \left( \frac{\theta_A}{2N} \right)^{2N} \exp \left[ -N \left( \frac{\theta_A - \theta_{\text{max}}}{\theta_{\text{max}}} \right)^2 \right] \]

Deductions:

\[ \left( \frac{\theta_A}{2N} \right)^{2N} \text{ is very big \# if } N \gg 1, \text{ multiplicity} \]

has an enormous peak

width of peak defined by when \( \theta_A - \theta_{\text{max}} \) cause \( 1/e \approx 1/3 \)

drop from maximum

\[ N \left( \frac{\Delta \theta}{2\theta_{\text{max}}} \right)^2 \sim 1 \Rightarrow \Delta \theta \approx \frac{\theta_{\text{max}}}{\sqrt{N}} \]

Full width half max \( = 2\Delta \theta \approx \frac{\theta_{\text{max}}}{\sqrt{N}} \)
Preparation For Calculating

Multiplicity of Atomic Ideal Gas: Gamma \( \Gamma^n \)

Read Appendix B.2 on Gamma \( \Gamma^n \)

Start with observation: For real \( a > 0 \),
\[
\int_0^\infty e^{-ax} \, dx = \left[ -\frac{e^{-ax}}{-a} \right]_0^\infty = \frac{1}{a} = \Gamma^{-1}
\]

Can differentiate both sides w.r.t. "\( a \)" (Theorem says this is ok)

So get:
\[
\frac{d}{da} \int_0^\infty e^{-ax} \, dx = \int_0^\infty \frac{d}{da} (e^{-ax}) \, dx = \frac{d}{da} (\Gamma^{-1})
\]
\[
= \int_0^\infty (-x) e^{-ax} \, dx = -a^{-2}
\]

Conclude:
\[
\int_0^\infty xe^{-ax} \, dx = a^{-2}
\]

Repeat \( n-1 \) more times
\[
\int_0^\infty x^2 e^{-ax} \, dx = 2a^{-3}
\]
\[
\int_0^\infty x^3 e^{-ax} \, dx = 6a^{-4}
\]
\[
\vdots
\]
\[
\int_0^\infty x^n e^{-ax} \, dx = n! a^{-n-1}
\]

Set \( a = 1 \) on both sides

\[
\text{Note: Differentiation of definite integral w.r.t. a parameter extremely powerful tricky, facts and easier than integrations by parts, keep it in mind}
\]
Euler's Gamma Function

Define \( \Gamma(x+1) = \int_0^\infty x^a e^{-x} \, dx \quad x \text{ a real } \# > 0 \)

This was Euler's insight of how to extend the idea of factorial to the real (even complex) numbers:

\[ x! = \Gamma(x+1) \]

definition of factorial for arbitrary \( x \)

For example:

\[ 0! = \Gamma(1) = \int_0^\infty e^x \, dx = 1 \]

\[ n! = \Gamma(n+1) \quad \text{we proved this on previous page for integer } n \]

In homework, you relate \( \Gamma\left(\frac{1}{2}\right) \) to Gaussian integral \( \int_0^\infty e^{-x^2} \, dx = \sqrt{\pi} \)

and conclude

\[ (-\frac{1}{2})! = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \]

you also show recursion relation similar to \( n! = (n-1)! \cdot n \)

\[ \Gamma(x+1) = x \cdot \Gamma(x) \quad \text{for real } \# > 0 \]

This lets us get other values

\[ \Gamma\left(-\frac{1}{2}\right) = \Gamma\left(-\frac{1}{2} + 1\right) = \Gamma\left(-\frac{3}{2} + 1\right) = - \]

\[ \frac{1}{2}! = \Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2} \cdot \sqrt{\pi} \]

\[ \frac{1}{2}! = \frac{\sqrt{\pi}}{2} \]

etc.

Gamma \( F \) is unique way to extend n! to real #s if one assumes \( \Gamma(1) = 1 \) and \( \Gamma(x+1) \) is convex.
Stirling's Formula Applied To

\[ \Gamma'(x) : \Gamma(x+1) \approx e^{-x} (x/e)^x \]

We derived Stirling's formula \( n! \approx \sqrt{2\pi n} (n/e)^n \) for integers \( n \) by writing \( \ln(n!) = \ln(1) + \ln(2) + \ldots + \ln(n) \) and approximating the sum by integral.

But we can prove result directly for Gamma function application of theorem that high powers of \( f^n \) look like Gaussian. Here is outline of proof (close to Schrodinger p. 320):

\[ \Gamma(x+1) = \int_0^\infty x^e e^{-x} dx = \int_0^\infty (xe^{-e/x})^x dx \]

\[ \Gamma(x+1) = x^{x+1} \int_0^\infty (e^{-y})^x dy \]

We have function \( f(y) = ye^{-y} \) has single global maximum at \( y = 1 \) (please verify) with value \( f_{\text{max}} = \frac{1}{e} \) \( f''(y_{\text{max}}) = -\frac{1}{e} \)

So:

\[ (ye^{-y})^x \approx \left( \frac{1}{e} \right)^x \exp \left[ -x(y-1)^2 \right] \]

\[ \Rightarrow \Gamma(x+1) \approx x \left( \frac{x}{e} \right)^x \int_0^\infty e^{-x(y-1)^2} dy \]

Let \( z = y-1 \)

\[ \approx x \left( \frac{x}{e} \right)^x \int_{-\infty}^{\infty} e^{-xz^2} dz = \sqrt{\frac{2\pi x}{e}} (\frac{x}{e})^x \text{qed.} \]
Multiplicty and Entropy of Monoatomic Ideal Gas

Am going to show that for N atoms of ideal gas in volume V
\[ S(z, N, V) = F(N) V N^{3N/2} \quad N \gg 1 \]

\[ F(N) = \left( \frac{2\pi m}{h^2} \right)^{3N/2} \frac{1}{N!(3N/2)!} \]

This will let us calculate the entropy \( S = k_b N \ln \Omega \) of atomic gas, giving valuable formula, Sackur-Tetrode equation:

Two observations:

(1) contains Planck's constant \( h \), so need quantum mechanics to get answer

(2) contains mass of atom \( m \) \( \Rightarrow \) entropy varies with \( m \), getting larger with bigger \( m \)

Will follow Schroedel's argument page 68-72

This is heuristic simplified argument that avoids more difficult and lengthy quantum derivation. Qualitative ideas are correct so useful.
\[ S^2(U, V, N) \propto U^N V^N \]

imply multiplicity sharply peaked for macroscopic system of two interacting ideal gases of some \( N \)

\[
\begin{array}{c|c|c}
\hline
U_A & U_B & U_A + U_B \\
\hline
V_A & V_B & V_A + V_B \\
\hline
\end{array}
\]

partition allows \( U_A, V_A \) to vary

\( U_A + U_B = U \)

\( V_A + V_B = V \)

\[ S_{\text{total}}(U_A, V_A) = S_A(U_A, V_A) S_B(U_B, V_B) \]

\[ = \left[ \frac{F(N)}{N} \right]^2 \left[ V_A \left( V - V_A \right) \right]^N \left[ U_A \left( U - U_A \right) \right]^{3N/2} \]

\[ \Delta V_A \sim \frac{V}{\sqrt{N}} \quad \Delta U_A \sim \frac{U}{\sqrt{3N/2}} \]

Exact same argument as Einstein solid

get enormous, extremely narrow peaks centered on equilibrium values

\[ V_A = \frac{V}{2} \quad U_A = \frac{U}{2} \]

extremely unlikely to observe anything but equilibrium values,

Challenge: can you work out details if \# of atoms can vary between \( A \) and \( B \)?

\[ S_{\text{total}}(N_A) = \frac{f(N_A)}{N_A} \frac{f(N-N_A)}{N-N_A} = \ldots \]

more complicated (too much so for this course)
Derivation of $S(N, V, N)$

For $N$ atoms of Ideal Gas in Volume $V$

1st question: what are microstates of system? need to count

atom described by six numbers $(x, y, z, p_x, p_y, p_z)$

location of center of mass $(x, y, z)$

momentum $\vec{p} = (p_x, p_y, p_z)$

I momentum but not velocity enters quantum mechanics so avoid using velocity $V$

Consider single atom $N=1$

gives $S_2(1) \propto V$

$V = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

energy conserved mean

$U = \frac{(\vec{p})^2}{2m} = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$

or $p_x^2 + p_y^2 + p_z^2 = (\sqrt{2mU})^2$

define surface of sphere of radius $r = \sqrt{2mu}$

guess that multiplicity proportional to surface area of sphere

$S_2(1) \propto V A_3(\sqrt{2mu})$

area of sphere in three dimensions