Assignment 1
Physics/ECE 176

Made available: Thursday, January 13, 2011
Due: Thursday, January 20, 2011, by the beginning of class.

Overview

Before beginning this assignment, please read carefully the part of the course syllabus:

`http://www.phy.duke.edu/~hsg/176/176-syllabus.html#grading`

concerning grading and homeworks so that you understand your responsibilities with respect to doing and handing in homework assignments. Assignments are due no later than the beginning of class on the due date.

Problem 1: An E-mail Introduction

Please introduce yourself by sending me an e-mail message (to `hsg@phy.duke.edu`) with the following information:

1. your major or intended major (if you know it);
2. your professional interests (if you know them);
3. where you grew up and what attracted you to Duke;
4. some of your interests and hobbies outside of classes;
5. anything about how you like courses taught that could help me teach a better course for you;
6. a brief list of the most advanced physics and math courses that you have taken (just the titles are fine). In particular, have you seen the material in Chapter 1 before so that I can go through that material quickly?

Note: Please send me this email introduction as soon as possible—preferably no later than Sunday, January 16—since your answer will help me to determine how rapidly I can move through Chapter 1. (Your answers will also help me to select examples to discuss later in the course.) The rest of this assignment is due on Thursday, January 20.

Problem 2: Learning to Use Mathematica

For this course, you are required to master the computer mathematics program Mathematica at the level of the tutorial mentioned below in part 2.4 so please work through this tutorial carefully. Part 2.4 also explains what you have to hand in for Problem 2.

1. First, install the latest version of Mathematica, version 8.0, on your computer. You can get a free copy for PCs, Macs, and Linux machines here

   `http://www.oit.duke.edu/comp-print/software/license/`

under the Software category "Statistics, Science, Mathematics, & Data Transfer".

2. After installing Mathematica, make sure you can actually run it. Start up Mathematica by double-clicking on the Mathematica icon. After a few seconds, a "Welcome to Wolfram Mathematica 8" window will appear. Click on the "Notebook" button in the upper left of this window to create a so-called notebook window. Inside the notebook window, type "1+2" and then type Shift-Enter to execute the command. (Shift-Enter means type the Enter key while holding down the Shift key.) You should get the answer "3". If you have difficulty installing and running Mathematica, please get help as soon as possible from a classmate, from Professor Greenside, or from OIT.

3. If you are new to Mathematica, go through the online tutorial "Hands-on Start to Mathematica" available at this URL:

   `http://www.wolfram.com/broadcast/screencasts/handsonstart`

4. Finally, go to the 176 homework webpage

   `http://www.phy.duke.edu/~hsg/176/homeworks`

   and download the Mathematica file `problem-1.17-mma-tutorial.nb` by right-clicking on the link and selecting the option "Save link as..."; your Desktop would be a convenient place to save the file. You can then open the file in Mathematica by simply double-clicking on the file icon. Once the file is opened in a notebook window, start reading through the tutorial and simply follow the instructions.

   What you have to hand in with this homework assignment is brief and is described in the second-to-last Mathematica cell labeled "Homework exercise: give me your best choice of a and b with a print-out of your best plot".

Problem 3: Physics Problems

1. Problem 1.6 on page 6 of Schroeder. Also explain briefly *why* you cannot feel accurately how hot or cold some object is for your specific example.

2. Problem 1.12 on page 8 in Schroeder. Make sure to explain how you obtained or estimated the size of a small molecule.

   For this problem, use a similar method to determine the average distance (in light years) between stars in a galaxy by using the Internet (or an astronomy textbook) to look up the typical number of stars in a typical galaxy and the typical size of a spiral galaxy.

   Also estimate the average distance between galaxies (in light years) based on the total number of known galaxies (about 10^{11}) in a universe of diameter approximately 14 billion light years.

   For all three answers, one significant digits is fine since there are many rough approximations being made here, e.g., neither stars nor galaxies are uniformly distributed in space.

3. Atoms and molecules are not crisp geometric objects with a precise size but fuzzy quantum mechanical things, with electron clouds that extend arbitrarily far from the nucleus. One way to associate a length with a molecule is its de Broglie (pronounced approximately as "de-broy", not as "de-brog-lee") wavelength $\lambda = h/p = h/(mv)$, which is related to the momentum $p = mv$ of the molecule. (See Eq. (A.3) on page 360 of the Schroeder text or your introductory physics and chemistry texts.) This wavelength is usually not the same size as the so-called *covalent radius*, which is defined to be one half of the distance between two nearest neighbor molecules in a crystalline solid as determined by X-ray crystallography (with the data extrapolated to absolute zero to eliminate the effects of thermal expansion).

For a given pressure, the average distance between molecules[1] $d \propto T^{1/3}$ decreases with decreasing temperature as a gas becomes denser and starts to turn into a liquid. On the other hand, the molecule's de Broglie wavelength[2] $\lambda \propto T^{-1/2}$ increases with decreasing temperature since the molecule is moving more slowly on average. An interesting question then arises: is it possible for the decreasing average distance between molecules in a gas to become about the same size as the increasing de Broglie wavelength? If so, does anything interesting happen physically?

Answer this question for a gas of He^4 atoms in a vessel of constant pressure $P = 1.0\,\mathrm{atm}$ whose volume can vary. For this problem, assume that the He gas is ideal.

(a) At what temperature T in kelvin do you predict that the de Broglie wavelength becomes equal to the average distance between molecules?

Note: you can use Eq. (1.21) on page 13 of Schroeder to estimate the average speed v of a He atom as a function of the gas's temperature T.

(b) What is the common value of these two lengths when they become equal? For comparison, the covalent radius of a He atom is about $3 \times 10^{-11}\,\mathrm{m} \approx 30\,\mathrm{pm}$ where $1\,\mathrm{pm} = 10^{-12}\,\mathrm{m}$ is one picometer.

(c) Discuss whether your estimate for T corresponds to any features in the phase diagram for He^4, as shown in the left panel of Figure 5.13 on page 168 of the Schroeder text.

Problem 4: Some Math Review

The following two problems are to help you start thinking about some mathematics that we will use later in the course.

1. Determine the value of the following triple integral in terms of the number π and some ratio of integers.

$$\int_0^\pi dx \int_0^\pi dy \int_0^\pi dz\, y^2 \sin(z)\,. \tag{1}$$

   It is not sufficient to write down the answer, you need to explain or show how you got the answer.

   A hint: if you think first conceptually about what this integral *means*, you can get the answer without evaluating any integrals[3]. But also make sure you know how to get the answer by carrying out the appropriate integrations.

2. Many times during this semester, we will use a trick that is widely used in science, engineering, and applied mathematics, which is to use the first few terms of the infinite Taylor series $f_0 + f_1(x-a) + f_2(x-a)^2 + \cdots$ of some function $f(x)$ about a point of interest $x = a$ to obtain a low-order polynomial that approximates the function well for locations x sufficiently close to the point a of interest. This often suffices to provide analytical insight about the properties of a complicated function, say how the heat capacity $C(T)$ of some substance behaves for small or large temperatures.

   Calculating the Taylor series of a function by hand can be tedious since it requires evaluating successive derivatives $f^n(a)$ of the function $f(x)$ at the point $x = a$ of interest. Fortunately, there are efficient ways to get the first few coefficients of the Taylor series for many functions that arise in physics by simple algebraic manipulations of polynomials, without evaluating any derivatives. To help you start thinking in this direction, please answer the following questions.

[1] Make sure that you know how to use the proportionality symbol $\propto$: we say that some quantity a is proportional to some quantity b, $a \propto b$, if there is a nonzero constant c such that $a = cb$. The notation is valuable when one wants to emphasize a functional relation without worrying about specific constants. The ideal gas law $PV = NkT$ implies $V \propto T$ for fixed N and fixed P and $V \approx Nd^3 \propto d^3$ which implies $d^3 \propto V \propto T$ which implies $d \propto T^{1/3}$ as claimed.

[2] If $p = mv$ is the momentum of a particle of mass m and speed v, the de Broglie relation $p = h/\lambda$ implies $\lambda \propto p^{-1} \propto v^{-1}$. But Eq. (1.21) on page 13 of Schroeder implies $v \propto T^{1/2}$ so $\lambda \propto v^{-1} \propto T^{-1/2}$ as claimed.

[3] Especially in a science course like 176, you should always try to think first about the possible meaning of some mathematical expression.

(a) You learned in high school about the infinite geometric series

$$1 + x + x^2 + x^3 + \cdots , \tag{2}$$

which converges to the function $(1-x)^{-1}$ for all numbers x such that $|x| < 1$. If the number x has a sufficiently small magnitude, just a first few terms of this infinite series can provide an accurate approximation to $(1-x)^{-1}$.

Show that Eq. (2) converges to $(1-x)^{-1}$ for all numbers x of magnitude less than one. Hint: first show that the sum $S_N(x)$ of the first N terms of this series is explicitly $(1-x^N)/(1-x)$.

(b) Show that Eq. (2) is in fact the Taylor series of the function $f(x) = (1-x)^{-1}$ about the point $x = 0$.

(c) In Eq. (2), if $|x|$ is sufficiently small, the first few terms of Eq. (2) can provide a useful approximation to $(1-x)^{-1}$ near $x = 0$. For example, we can use the first three terms of the Taylor series Eq. (2) to approximate $(1-x)^{-1}$ in the vicinity of $x = 0$ by the quadratic $1 + x + x^2$:

$$(1-x)^{-1} \approx 1 + x + x^2, \qquad |x| \ll 1, \tag{3}$$

and this approximation becomes arbitrarily accurate[4] for sufficiently small $|x|$.

Further, if instead of thinking of x as some number we think of x more broadly as some small arbitrary mathematical expression, Eq. (3) provides a way to replace a reciprocal in some expression by multiplication with a low-order polynomial, and polynomials are often easier and more insightful to work with.

To appreciate this point—replacing a reciprocal by multiplication by some low-order polynomial—**use Eq. (3) followed by multiplication of polynomials to show that the first three terms of the Taylor series of the function**

$$f(x) = \frac{1 + x^2}{1 + x - 2x^2}, \tag{4}$$

is given by the quadratic

$$1 - x + 4x^2. \tag{5}$$

Note that you have obtained these terms of the Taylor series[5] without any tedious differentiations of Eq. (4).

Here are some hints:

i. Write $f(x)$ in the form $(1 + x^2) \times (1 - [-x + 2x^2])^{-1}$ and observe that the expression in brackets is small if x is sufficiently small in magnitude. You can then use Eq. (3) with x on both sides replaced by $-x + 2x^2$.

ii. The math is easy because you can just throw out (set to zero) any power higher than two that appears since these higher powers are acceptably tiny compared to lower powers that you keep.
For example, the product $(1 + x^2 - x^5)(1 - x + x^2)$ for $|x|$ sufficiently small can quickly be seen to equal the polynomial $1 - x + 2x^2$ up to powers of x^2. First, the x^5 term in the first factor $1 + x^2 - x^5$ can be ignored right away as tiny compared to x^2 so the first factor becomes $1 + x^2$. Second, when the x^2 in the first term multiplies the second term $1 - x + x^2$ to give $x^2 - x^3 + x^4$, all powers except the first can be ignored as small compared to x^2, i.e., $x^2 - x^3 + x^4 \approx x^2$ for small enough x. Thus: $(1+x^2-x^5)(1-x+x^2) \approx (1+x^2)(1-x+x^2) = 1 \times (1 - x + x^2) + x^2(1 - x + x^2) \approx 1 - x + x^2 + x^2$.

[4] For the mathematically inclined, try showing that no other quadratic polynomial approximates $f(x)$ near $x = 0$ as well as the first three terms of its Taylor series.

[5] Can you figure out why Eq. (5) is indeed the first three terms in the Taylor series about $x = 0$ of the function $f(x)$?

(d) To verify my claim that the first few terms of a Taylor series provides a good approximation to a function near the point of expansion, use the following Mathematica code to explore visually the accuracy of approximating Eq. (4) by Eq. (5). **Please print out and include in your homework the plot g2 of the relative error. Also explain briefly what you learn from your two plots.** For example, over what range of x does the second-order approximation give a relative error smaller than 10% (corresponding to one significant digit of accuracy)? Smaller than 1% (two significant digits)?

```
f[x_]  :=  ( 1  +  x^2 ) / ( 1  +  x  -  2x^2 ) ;
p[x_]  :=  1  -  x  +  4 x^2 ;

g1 = Plot[
  { f[x], p[x] } ,       (* plot f and p simultaneously on same plot *)
  { x, -0.5, 0.5 } ,     (* plot over range -0.5 <= x <= 0.5 *)
  PlotRange -> { 0, 5 }  (* plot function over vertical range [0,5] *)
]

g2 = Plot[         (* plot relative error |f(x) - g(x)|/|f(x)| *)
  Abs[ (f[x] - p[x]) / f[x] ] ,
  { x, -0.25, 0.25 }   (* plot over range -0.25 <= x <= 0.25 *)
]
```

Problem 5: Time to Finish This Homework Assignment

Please tell me the approximate time in hours that it took you to complete this homework assignment.