1. To two significant digits,
\[ k \approx 1.4 \times 10^{-23} \, \text{J/K} \]

Please memorize this value over the semester, I do not plan to provide it in future exams or quizzes.

The units can be deduced in several ways; easiest is to use equipartition for an atomic ideal gas:

\[ \langle \frac{1}{2} m v^2 \rangle = \frac{3}{2} kT \]

units of energy \( \leftrightarrow \) units of Kelvin or Joule

Crucial that \( T \) be an absolute temperature for this expression to make sense. \( T=0 \) is absolute zero.

2. For an Einstein solid with \( N \) identical oscillators and \( q \) units of energy, if \( N \gg q \gg 1 \), this means that the number of energy units per oscillator \( q/N \) on average will be a small number
so most oscillators will be in their lowest energy state, or ground state. But the lowest possible energy is \( g=0 \), all oscillators in their ground state, and this must correspond to absolute zero \( T=0 \).

So, \( N \gg g \gg 1 \) implies the Einstein solid must be close to absolute zero, a low-temperature regime.

\[ \text{(2b)} \]

The discussion is exactly similar to pages 63-64 of Schroedinger and to what I discussed in lecture. As long as you sweated these details, you should have been able to answer this question.

In fact, if you remembered or knew that the high temperature limit \( g \gg N \gg 1 \) led to the approximation

\[
S_L = \left( \frac{g+N-1}{g} \right) \approx \frac{(g+N)!}{g!N!} \approx \left( \frac{e^g}{N} \right)^N
\]

and realized that \( S_L = \frac{(g+N)!}{g!N!} \) is symmetric with respect to the symbols \( g \) and \( N \), you would have
Quix-3 Answer

realized that the answer to 2b was obtained by simply swapping the symbols \( q \) and \( N \)

\[
\left( \frac{eN}{q} \right)^N \xrightarrow{N \to \infty} \left( \frac{qe}{N} \right)^N \quad \text{for } q/N \gg 1
\]

But I did want you to show me that you mastered the key technical details:

1. take \( \ln \) to simplify the math
2. use Stirling to simplify the factorials
   \[ K(N!) \approx n \ln n - n \]
3. Look for terms of form \( \ln (N+q) \approx \ln N + \frac{q}{N} \)
   since \( q/N \ll 1 \)

Too many students avoided using a \( \log \) and ended up with expressions like

\[
SC(N,q) \approx \frac{\sqrt{2\pi(N+q)} \left( \frac{N+q}{e} \right)^{N+q}}{\sqrt{2\pi q} \left( \frac{q}{e} \right)^q \sqrt{2\pi N} \left( \frac{N}{e} \right)^N} \approx \left( \frac{N+q}{q} \right)^q \left( \frac{N+q}{N} \right)^N
\]

and then said: "oh, \( N \gg q \) so let's replace \( N+q \approx N \)"

which would give

\[
SC(N,q) \approx \left( \frac{N}{q} \right)^q
\]

but then what does the \( e^q \) term come from? Key point
is that one has to retain small quantities of order \( \frac{g}{N} \) by a perturbation or series approximation and this leads to the \( e^g \) factor.

If you remember your intro calculus, that

\[
\lim_{N \to \infty} \left(1 + \frac{g}{N}\right)^N = e^g \tag{41}
\]

you could have gotten the answer without using \( \ln 52 \) but you derive (41) by taking logs:

\[
y = \left(1 + \frac{g}{N}\right)^N
\]

\[
\ln y = N \ln \left(1 + \frac{g}{N}\right) \approx N \left(\frac{g}{N}\right) = g \quad \text{if} \quad g \ll N
\]

\[
\Rightarrow \quad y \approx e^g \quad \text{for big } N
\]

Best bet: take log of an expression when dealing with a very large number.

Starting with \( S(N; g) \approx \left(\frac{eN}{g}\right)^g \) and using \( g = \epsilon g \), we would proceed as follows to get the heat capacity \( C = \frac{dU}{dT} \)
\[ S = k \ln S_2 = k \ln \left[ \frac{(eN)^2}{\bar{q}} \right] \]

\[ = k \bar{q} \left[ \ln (N\bar{e}) - \ln (\bar{q}) \right] \]

Now get temperature from \( \frac{1}{T} = \frac{2S}{k} \): \(
\frac{1}{kT} = \frac{2 \ln S_2}{2k} = \frac{2 \ln S_2}{2k} \frac{2k}{2k} = \frac{1}{e} \frac{2 \ln S_2}{2k} \)

\[ = \frac{1}{e} \frac{2k}{2k} \left[ \theta \left( \ln (N\bar{e}) - \ln (\bar{q}) \right) \right] \]

\[ = \frac{1}{e} \left[ \ln (N\bar{e}) - \ln (\bar{q}) + \theta \left( -\frac{1}{\theta} \right) \right] \]

\[ = \frac{1}{e} \left[ \ln N + \ln e - \ln q - 1 \right] \]

\[ \frac{1}{kT} = \frac{1}{e} \left[ \ln N - \ln \bar{q} \right] = \frac{1}{e} \ln \left[ \frac{N}{\bar{q}} \right] = \frac{1}{e} \ln \left[ \frac{N\bar{e}}{\bar{q}} \right] \]

This can be rewritten to solve for \( U = U(T) \): \( \frac{e}{kT} = \ln \left( \frac{N\bar{e}}{\bar{q}} \right) \Rightarrow e^{1/kT} = \frac{N\bar{e}}{\bar{q}} \)

or \( U = (N\bar{e})e^{-e/kT} \)

Note: details of this expression that make sense: left side is energy, right side has units \( NE \) of energy.
and argument of exponential $\frac{e}{kT}$ must be dimensionless, which it is since $e$ has units of energy so does $kT$.

Final step is just to differentiate to get $C$:

$$C = \frac{dN}{dt} = N e \frac{d}{dt} \left( e^{-e/kT} \right) = N e \left[ e^{-e/kT} \times \frac{d}{dt} \left( e^{-e/kT} \right) \right]$$

by chain rule

\[
C = N k \cdot \left( \frac{e}{kT} \right)^2 \cdot e^{-e/kT}
\]

(\times 1)

which was the desired answer. This agrees with what you did in a recent homework problem and agrees with the low-$T$ limit of the full expression

$$C = N k \left( \frac{e}{kT} \right)^2 \frac{e^{e/kT}}{(e^{e/kT}-1)^2}$$

for an Einstein solid, valid for all values of $T$.

You should be able to verify, say by L'Hôpital's rule, that

$$\lim_{T \to 0} \left( \frac{e}{kT} \right)^2 e^{-e/kT} = 0$$

\[\frac{1}{T}\]
Multiple choice I: answer is (d), \((A_3(R))^N A_{2N}(\text{emu})\)

Reason is following: if gas is ideal, molecules don't interact with one another so multiplicity will be proportional to "space available to particle 1" \(\times\) "space available to particle 2" \(\times\) \(\ldots\) \(\times\) "space available to particle \(N\)".

But if particles move on sphere of radius \(R\), the available space is the surface area \(A_3(R)\). So we conclude

\[S^2 \propto [A_3(R)]^N A_{2N}(\text{emu})\]  
if gas is ideal

which rules out answers (a), (c), and (e).

Next important insight is that particles are moving on surface of sphere, which is a two-dimensional domain. So momentum vectors \((p_x, p_y, p_z)\) of given particle, although a 3-vector, is constrained to be tangent to the sphere, so there are only two independent components. We conclude

\[S^2 \propto [A_3(R)]^N A_{2N}(\text{emu})\]  
because surface of sphere is 2D, reduced dimensionality.
True False

1. T/F is False, see middle of page 57 of Schroeder

2. False This was key point of discussion on pages 57-59 of Schroeber. In an isolated Einstein solid consisting of two subsystems A and B, if A has energy $E_A$ and B has energy $E_B$, then over short times (compared to a thermal relaxation time $E/k$), the multiplicity of the entire system is given approximately by

$$S_{total} = S_A(E_A)S_B(E-E_A)$$

But over longer times, when equilibrium is attained, the total multiplicity can be much bigger:

$$S_{total} = \left(\frac{N_A+N_B+E_A+E_B}{E_A+E_B} - 1\right)$$

The total multiplicity can and does change with time (unless one started already in equilibrium) and increases steadily until the multiplicity reaches a maximum. This is in fact the second law of thermodynamics, $dS/dt > 0$, expressed in terms of the multiplicity $S_C$. 
True-False

3. (False) The Sackur-Tetrode equation given in the quiz,

\[ S = Nk \left[ \frac{5}{2} + \ln \left( \frac{V}{N} \left( \frac{y \mu m}{3h^2 N} \right)^{3/2} \right) \right] \]

depends on the mass of the atoms in the gas. In fact,

\[ S = \frac{3}{2} Nk \ln(m) + \text{other stuff not dependent on } m \]

So increasing the atomic mass directly increases the entropy since \( \ln(x) \) is a monotone increasing function of \( x \). Argon has a greater entropy than He.

4. (True) This was a point made in lecture and you were asked in a homework problem, Problem 2.7 of Assignment 5, to show directly that you could apply Einstein’s theory of a solid to the vibrational mode of a diatomic gas. In fact, Einstein’s theory is more correctly applicable to the diatomic molecules than to crystalline vibrations since the molecules are exactly identical, while oscillators in a solid have a continuous range of frequencies and so are neither identical nor independent.
True-False

5. **False** By expanding the log in the Schur-Tetrode equation given in the quiz, you can isolate the contributions of the variables \( U, V, \) and \( N, \)

\[
S = Nk\ln V + \frac{3}{2} Nk\ln U - \frac{5}{2} N\ln N + \text{stuff}
\]

where "stuff" doesn't depend on \( V, U, \) or \( N. \) From this form, we see immediately that making \( V \) sufficiently small and positive or making \( U \) sufficiently small and negative can make \( S \) negative since \( \ln(x) \to -\infty \) as \( x \to 0 \):

\[
\begin{align*}
&\ln(1) \\
&\downarrow \\
&x \\
&\leftarrow \log \text{ diverges to } -\infty \text{ as } x \to 0^+
\end{align*}
\]

This is unphysical: since \( S \geq 1, \) \( k\ln S = S \geq 0. \) The more physical criterion for Schur-Tetrode giving a negative value comes from writing

\[
S = Nk \left[ \frac{5}{2} + \ln \left( \frac{V/N}{\lambda^2} \right) \right]
\]

\[
\lambda = \frac{h}{\sqrt{2}\mu kT}
\]

If \( T \) (here \( U \)) is small enough or \( V/N \) is small enough, \( V/N \ll \lambda^2 \) and \( S \) becomes negative.
True-False

6. 

(-1)! = \infty I graded this as true but the answer is strictly \( \text{false} \), so please get in touch with me if you got this T/F marked wrong, so I can restore 2 points to your score.

First, I didn’t expect you to have to do any calculation here, I was hoping that, as you read Schroeder’s Appendix B.2 about the Gamma function, you had looked at the plot of \( \Gamma(x) \) vs \( x \) in Fig. B.3 on page 388 and noticed that

\[ |(-n)!| = \infty \]

For every positive integer \( n \geq 1 \), i.e., \( |(-n)!| \) have infinite magnitude. (I had left out the absolute value sign for the quiz, so \((-1)! = \infty \) or \(-\infty\) are acceptable answers.) The actual value of \((-n)!\) for \( n \) a positive integer depends though from what side you take the limit \( x \to -n \), from the left or right.
To understand what is going on, we can look at the formal value of \( x! \) for \( x \) any negative real number.

\[
x! = \Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = \int_0^\epsilon t^x e^{-t} dt + \int_\epsilon^\infty t^x e^{-t} dt \quad x < 0
\]

Here I have done a common trick for an integral that has a divergent integrand \( t^x e^{-t} \to \infty \) as \( t \to 0 \) if \( x < 0 \).

I split the integral into a part close to the point of divergence \( \int_0^\epsilon \), where \( \epsilon \ll 1 \) is a tiny number, and a part \( \int_\epsilon^\infty \) that is finite. For sufficiently small \( \epsilon \), \( e^t \ll 1 - t \ll 1 \) since \( t \ll 1 \) between \([0, \epsilon]\).

So,

\[
\int_0^\epsilon t^x e^{-t} dt = \int_0^\epsilon t^x dt = \frac{t^{x+1}}{x+1} \bigg|_0^\epsilon = \frac{\epsilon^{x+1}}{x+1} - \lim_{\epsilon \to 0} \frac{\epsilon^{x+1}}{x+1}
\]

For \( x < -1 \), this diverges to \(-\infty\); for \( x = -1 \), the integral becomes \( \int_0^\epsilon t^{-1} e^{-t} dt = \int_0^\epsilon \frac{dt}{t} \approx \ln \epsilon - \lim_{\epsilon \to 0} \ln \epsilon \to -\infty \), while for \(-1 < x < 0\), the integral is finite.

So strictly speaking, \( x! \) diverges to \(-\infty\) for \( x \leq -1 \) based on the integral. But mathematicians long ago observed that, while the integral diverges, one could get a finite value of \( x! \) for any \( x \) not a negative integer by using the recursion relation:

\[
\Gamma(x+1) = x! \quad x \in \mathbb{R}, x \neq -1, -2, -3, \ldots
\]
\[\Gamma(x) = \frac{1}{x} \Gamma(x+1)\]

repeatedly until one was evaluating \(\Gamma\) at positive values. For example

\[\Gamma(-\sqrt{2}) = \Gamma(-1,4) = \frac{1}{(-\sqrt{2})} \Gamma(1-\sqrt{2})\]

\[= \frac{1}{(-\sqrt{2})} \frac{1}{(1-\sqrt{2})} \Gamma(2-\sqrt{2})\]

 Finite positive value

Since \(2-\sqrt{2} > 0\)

For \(x\) negative and non-integer, let \(m\) be the integer closest to but larger than \(x\), \(x < m < 0\). Then

\[\Gamma(x) = \frac{1}{x} \Gamma(1+x) = \frac{1}{x(1+x)} \Gamma(2+x) = \ldots = \frac{1}{x(1+x) \ldots (m+x)} \Gamma(m+1+x)\]

Again, we end up with a finite value since \(x\) is not an integer and \(\Gamma(m+1+x)\) is evaluated at a positive value. But now you can see why \(x!\) diverges to \(+\infty\) for \(x\) a negative integer, we can take the limit \(x \to -k^+\) or \(x \to -k^-\) and the factor \(\frac{1}{x+k}\) will diverge to \(+\infty\). So even with a definition in terms of the recursion relation, one can not avoid \((-n)!\) diverging for \(n > 0\).