**Print your name clearly:**________________________________________

**Signature:**________________________________________

"I agree to neither give nor receive aid during this exam."

# Physics 176 Final Exam

Professor Greenside
Thursday, May 6, 2010

Please read the following carefully before starting the test:

1. This exam is closed book and will last the entire exam period.
2. No calculators or other electronic devices are allowed.
3. Please hand in your two pages of notes with your exam (and put your name on the two pages).
4. Look over the entire exam and get a sense of its length, what kinds of questions are being asked, and which questions are worth the most points.
5. Answer the true-false and multiple choice questions on the exam itself. Answer all other questions on the extra blank pages. If you need extra pages during the exam, let me know.
6. Please write your name and the problem number at the top of each extra page.
7. Please write clearly. If I can not *easily* understand your answer, you will lose credit.
8. Unless otherwise stated, you must justify any written answer with enough details for me to understand what you are doing.
9. If you are not sure about the wording of a problem, please ask me during the exam.

## Problems That Require Writing

Please write your answers to the following problems on extra blank sheets of paper. Also make sure to write your name and the problem number at the top of each sheet. Unless otherwise stated, *you need to justify your answers to get full credit.*

1. **(10 points)** Sketch qualitatively correct graphs of the Gibbs free energy $G = U - TS + PV$ versus temperature $T$ for the three phases of water (ice, water, and steam) at atmospheric pressure. You should draw your three graphs on the *same* set of axes so that you can see how they imply which phase is stable at a given temperature. Also make sure to indicate where on your temperature axis $T = 0^\circ$ C and $T = 100^\circ$ C.

2. **(20 points)** Consider the following 3-step cyclic process $A \to B \to C \to A$ in the pressure-volume plane that characterizes an ideal monoatomic gas: the gas starts at point $A = (V_0, P_0)$ with initial temperature $T_0$, initial pressure $P_0$ and initial volume $V_0$. The gas then expands isobarically to point $B = (2V_0, P_0)$, is then compressed by following a straight line segment from point $B$ to the point $C = (V_0, 2P_0)$, and finally the gas is brought back to the the point $A$ by an isochoric process. After this cycle is carried out once, determine

   (a) the total change in energy $\Delta U$ of the gas;
   (b) the total heat $Q$ added to the gas;
   (c) the total work $W$ done on the gas;
   (d) the total change in temperature $\Delta T$ of the gas;
   (e) the total change in entropy $\Delta S$ of the gas.

   Assume that each step is carried out quasistatically (the gas and environment are always in thermodynamic equilibrium).

3. **(10 points)** Estimate to the nearest power of ten how many candy bars you would have to eat during a 24-hour period to supply the energy that you lose to the surrounding environment via blackbody radiation from your skin. To simplify this problem, assume that during this 24-hour period you are floating in outer space without clothes so that no heat is returned to your body by clothes, reflection, or by surrounding air, and assume that your skin is a perfect blackbody emitter (emissivity $e = 1$).

   Note: a typical candy bar provides about 250 Calories, one Calorie is about 4,200 J, and the Stefan-Boltzmann constant has the value $\sigma \approx 6 \times 10^{-8}\, \mathrm{W/(m^2\, K^4)}$.

4. **(15 points)** In one of the other universes of the multiverse, there is a particle (let's call it a mirron) that obeys the laws of quantum mechanics but unlike a boson or fermion, a mirron has the property that, for any single-state quantum energy level $\epsilon$, there can be 0, 1, or 2 mirrons in that energy level. Mirrons have the further properties that their total number is not conserved and that each mirron can exist in three distinct polarization states. If a finite volume $V$ of mirrons is in thermodynamic equilibrium with temperature $T$ and if mirrons interact weakly so that they form an ideal gas, determine the energy distribution function $\mathcal{D}(\epsilon)$ for mirrons (i.e., the amount of energy contributed by mirrons whose energies lie in the range $[\epsilon, \epsilon + d\epsilon]$). Determine also how the pressure $P$ and heat capacity $C_V$ of a mirron gas vary with the temperature $T$ (your answers here will be simple powers of $T$).

   Note: In a cubic box of volume $V = L^3$, the quantum states of this particle are labeled by positive integers $n_x$, $n_y$, and $n_z$, and the energy of a given mirron state is given by $\epsilon(n_x, n_y, n_z) = \epsilon(n) = \alpha n^2/L^2$ where $\alpha > 0$ is a constant, $L$ is the size of the cubic box, and $n = \sqrt{n_x^2 + n_y^2 + n_z^2}$.

5. **(10 points total)** Consider a magnetic substance of volume $V$ that responds to the presence of a magnetic field $\mathbf{B}$ of strength $B$ *inside* the substance by becoming magnetized along the direction of $\mathbf{B}$ with a magnetization of magnitude $M$. In studying the thermodynamics of magnets, it turns out that a key variable is an auxiliary magnetic field $\mathcal{H}$ defined by

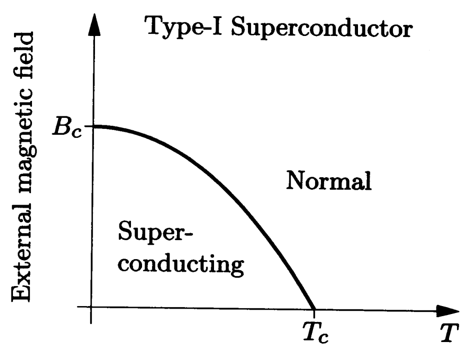
$$\mathcal{H} = \frac{1}{\mu_0} B - \frac{M}{V}, \tag{1}$$

where $\mu_0$ is the vacuum permeability. It then turns out that the thermodynamic potential $G_m$ that is minimized when the magnetic system is in thermodynamic equilibrium for constant temperature $T$ and constant field $\mathcal{H}$ is a magnetic analog of the Gibbs free energy that satisfies the following thermodynamic identity:

$$dG_m = -S\,dT - \mu_0 M\,d\mathcal{H}. \tag{2}$$

(a) **(5 points)** Derive an analogue of the Clausius-Clapeyron relation for the slope of a phase boundary in the $\mathcal{H}-T$ plane of this magnetic substance. You should write your equation in terms of the difference in entropy of the two phases.

(b) **(5 points)** When certain metals are cooled to a sufficiently low temperature in the presence of an external magnetic field, the metal can become a so-called type-I superconductor in which the resistance decreases enormously. (Superconducting wires are used in the magnets at the Large Hadron Collider and have been proposed for use in national power grids, to transport electricity over long distances without loss.) A representative phase diagram for a type-I superconductor is given by the following figure




In such a superconductor, surface currents flow in such a way so as to completely cancel the magnetic field inside (that is $B = 0$ but $\mathcal{H}$ is not zero in Eq. (1)). Given that the magnetization $M$ of the metal is essentially zero in its normal state ("normal" means "non-superconducting"):

i. Use your magnetic version of the Clausius-Clapeyron equation to determine which phase has the greater entropy, superconducting or normal.

ii. Determine what are the differences in entropy of the two phases at the two end points of the superconducting-normal phase line.

6. **(15 points)** During the semester, we discussed how to calculate the fractional coverage $\theta$ of a surface that was in equilibrium with a surrounding ideal gas of identical atoms of mass $m$ that had a fixed temperature $T$ and fixed chemical potential $\mu$. We first assumed that the surface resembled an egg carton with specific fixed locations where an atom could adsorb with binding energy $\epsilon_s < 0$. (The subscript $s$ means "surface".) We then calculated the grand partition function $\mathcal{Z}$ for the surface, from which we were able to calculate the average number of occupied sites, which then gave the coverage.

Solve this problem again—calculate the surface particle density $n_s = N_s/A$ as a function of the external gas pressure $P$ where $N_s$ is the number of adatoms and $A$ is the surface area—but now make a different assumption about the surface: instead of having specific binding sites like an egg carton, assume that the surface is perfectly smooth so that, once an atom adsorbs (again with a binding energy $\epsilon_s < 0$), the adatom can glide around as a free particle and all the adatoms together form a two-dimensional ideal gas in a finite surface area $A$.

Is your answer again a Langmuir isotherm, with $n_s \propto P/(P_0 + P)$ where $P_0$ is some constant? That is, does the behavior of surface coverage with pressure depend on whether adatoms bind in fixed locations or are free to move about?

Two hints: thermodynamic equilibrium between the two-dimensional surface gas and surrounding gas requires that their chemical potentials be equal, and the energy of an adatom can be written in the form $E = \epsilon + (p_x^2 + p_y^2)/(2m)$ (if we assume that the surface is the $xy$ plane of some coordinate system).

7. **(35 points total)** As a simple model of a so-called antiferromagnet, consider three identical spin-1/2 magnetic dipoles that are placed at the coordinates $(x, y) = (-1, 0)$, $(0, 0)$, and $(1, 0)$ of an $xy$ Cartesian coordinate system. The entire system is immersed in a uniform magnetic field $\mathbf{B} = B\hat{y}$ of strength $B$ that points in the positive $y$ direction of the coordinate system. Unlike the dipoles of a paramagnet, these magnetic dipoles are so close to one other that adjacent dipoles interact (but not dipoles that are not nearest neighbors). If we describe the state of each dipole by a spin variable $s_i = \pm 1/2$ that has the value $+1/2$ if the spin is up (points in the $\hat{y}$ direction) and has the value $-1/2$ if the spin is down, the energy of a particular state of this system can be written in the form:

$$E(s_1, s_2, s_3) = J\,(s_1 s_2 + s_2 s_3) - 2\mu B\,(s_1 + s_2 + s_3)\,. \tag{3}$$

The constant $J > 0$ is called a "coupling constant" that measures how strongly one spin couples to its neighbor, and a positive $J$ favors antiparallel nearest neighbors. The term $-2\mu B(s_1 + s_2 + s_3)$ is the one you have seen before in our discussion of a paramagnet.

(a) **(10 points)** Summarize in a table with several columns all the microstates of this system. For each microstate, give its energy for general values of $J$ and $B$. Then list the degeneracy of the states for the two cases of zero external magnetic field $B = 0$, and for a small external magnetic field such that $0 < B \ll J/\mu$ (nearest neighbor interactions are much stronger than the interaction of each spin with the external magnetic field).

(b) **(10 points)** Assume now that this spin system is allowed to reach thermodynamic equilibrium by placing it contact with a thermal reservoir with constant temperature $T$. For each of the following three conditions, deduce and give the values of the energy $U$, the entropy $S$, and the magnetization $M(T, B) = 2\mu(s_1 + s_2 + s_3)$ (so nine numbers in all).

i. $T = 0$ and $B = 0$.

ii. $T = 0$ and $0 < B \ll J/\mu$.

iii. $B = 0$ and $kT \gg J$.

(c) **(15 points)** For the case of zero external magnetic field ($B = 0$), deduce and sketch how the heat capacity $C(T)$ of this system varies with temperature for $T \geq 0$. Also calculate the approximate functional behavior of $C(T)$ for low temperatures ($kT/J \ll 1$) and for high temperatures ($kT/J \gg 1$).

Note: "approximate functional behavior" in some limit means carrying out some kind of Taylor series approximation to the lowest-order nontrivial term. You can also be efficient by avoiding an explicit calculation of $C(T)$ (which is a bit unwieldy). Instead, figure out qualitatively how the energy $E(T)$ varies with temperature and also deduce the functional forms of $E(T)$ for small and large $T$. You can then differentiate those limiting expressions of $E$ to get the limiting behaviors of $C(T)$.

## True or False Questions (2 points each)

For each of the following statements, please circle **T** or **F** to indicate whether a given statement is true or false respectively.

1. **T / F** It is possible for the phase line separating a crystalline solid phase from a liquid phase in a temperature-pressure phase diagram to end abruptly in a critical point.

2. **T / F** A system can be in thermodynamic equilibrium in the presence of a time-independent but spatially varying electric field.

3. **T / F** A gas of $N$ identical particles is ideal if and only if the single particle partition function $Z_1$ satisfies $Z_1 \gg N$.

4. **T / F** At a low temperature of $10^{-3}\,\mathrm{K}$, metals have a higher heat capacity than insulators.

5. **T / F** The root-mean-square variation $\Delta n(f)$ of the number of photons in an equilibrium photon gas that have frequency $f$ (i.e., are in the energy level $\epsilon = hf$) is smaller than the average number $\overline{n}(f)$ of photons that have frequency $f$.

6. **T / F** In a temperature-pressure phase diagram, the solid-gas phase transition line always passes through the origin $P = T = 0$.

7. **T / F** If three tiny holes are punched in the sides of a tall vertical enclosed cylinder near the cylinder's bottom, middle, and top, and if the cylinder contains an ideal gas in thermodynamic equilibrium, then the loss of gas by effusion will occur at the same rate for all three holes.

8. **T / F** If two identical blocks of metal are welded to form a single larger metal block, the Fermi energy $E_F$ will now be larger.

9. **T / F** The chemical potential $\mu$ of an ideal gas is zero if and only if the particles that make up the gas have zero mass.

10. **T / F** When a star supernovas and most of its mass collapses into a black hole (which is then characterized by just its mass $M$, charge $Q$, and angular momentum $\mathbf{\Omega}$), the entropy of the hole is much less than the entropy of the original star.

11. **T / F** For $N$ of order Avogadro's number, $\ln[(N!)!] \approx N! \ln(N)$.

12. **T / F** For an equilibrium low-density ideal gas that consists of $N$ identical molecules, the single-particle partition function $Z_1$ is an extensive variable.

13. **T / F** In a universe with ten spatial dimensions (a possibility suggested by string theory), the heat capacity $C_V$ of an equilibrium ideal gas consisting of $N$ identical atoms with temperature $T$ has the same value $(3/2)Nk$ that the same gas would have in our three dimensional universe.

14. **T** / **F** The temperature dependence of the pressure of a photon gas does not depend on whether the gas is three-dimensional or two-dimensional.

15. **T** / **F** There are 130 distinct ways to place three identical bosons in ten degenerate energy levels.

16. **T** / **F** The equilibrium temperature of the Earth due to absorption of sunlight and blackbody emission depends on the radius of the Earth.