

Print your name clearly:_____

Signature:_____

“I agree to neither give nor receive aid during this exam.”

Final Exam for Physics 176

Professor Greenside
Wednesday, April 29, 2009

This exam is closed book and will last the entire exam period. Please note the following:

1. Please hand in your 1-page of notes with your exam.
2. Before answering any questions, take a moment to look over the entire exam and get a sense of its length, what kinds of questions are being asked, and which questions are worth the most points.
3. True-false and multiple choice questions should be answered on the exam itself.
4. All other questions should be answered on the extra blank pages. If you need extra pages during the exam, let me know.
5. Please write your name and the problem number at the top of each extra page.
6. Please write clearly. If I can not easily understand your answer, you will lose credit.
7. Unless otherwise stated, you must justify any written answer with enough details for me to understand what you are doing.
8. Please feel free to ask me questions during the exam if the wording of a problem is not clear.

The following data and formulas may be useful:

$$k \approx 1.4 \times 10^{-23} \frac{\text{J}}{\text{K}}, \quad h \approx 6.7 \times 10^{-34} \text{ J} \cdot \text{s}, \quad g \approx 10 \frac{\text{m}}{\text{s}^2}, \quad N_A \approx 6.0 \times 10^{23} \frac{\text{molecules}}{\text{mole}}. \quad (1)$$

$$PV = NkT, \quad \Delta U = Q + W, \quad U = Nf(kT/2), \quad dU = TdS - PdV + \mu dN. \\ \Omega(n, q) = \binom{n+q-1}{q}, \quad x! = \Gamma(x+1) \approx \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \text{ for } x \gg 1. \quad (2)$$

$$S = k \ln \Omega, \quad S = \frac{Q}{T}, \quad \Delta S = \int_{T_1}^{T_2} \frac{C(T)}{T} dT. \quad (3)$$

$$F = U - TS, \quad dF = -SdT - PdV + \mu dN, \quad Z = \sum_s e^{-\beta U(s)}, \quad F = -kT \ln(Z). \quad (4)$$

$$G = U - TS + PV, \quad dG = -SdT + VdP + \mu dN, \quad G = N\mu. \quad (5)$$

$$\Phi = U - TS - \mu N, \quad d\Phi = -SdT - PdV - Nd\mu, \quad \mathcal{Z} = \sum_s e^{-\beta[U(s) - \mu N(s)]}, \quad \Phi = -kT \ln(\mathcal{Z}). \quad (6)$$

$$Z = \frac{1}{N!} Z_1^N, \quad Z_1 = \frac{V}{V_Q} Z_{\text{int}}, \quad V_Q = \left(\frac{h}{\sqrt{2\pi m k T}} \right)^3, \quad \mu_{\text{gas}} = -k T \ln \left(\frac{V Z_{\text{int}}}{N V_Q} \right). \quad (7)$$

$$p_n = \frac{h}{\lambda_n} = \frac{h n}{2L}, \quad \epsilon_{\text{electron}} = \frac{p^2}{2m} = \frac{h^2}{8mL^2} (n_x^2 + n_y^2 + n_z^2). \quad (8)$$

$$\epsilon_F = \frac{h^2}{8m} \left(\frac{3N}{\pi V} \right)^{2/3}, \quad U = \frac{3}{5} N \epsilon_F, \quad P = \frac{2}{3} \frac{U}{V}, \quad T_D = \frac{h c_s}{2k} \left(\frac{6N}{\pi V} \right)^{1/3}. \quad (9)$$

$$I = \frac{\mathcal{L}}{4\pi d^2}, \quad \mathcal{L} = (4\pi R^2) \times \sigma T^4. \quad (10)$$

$$\left(\frac{dP}{dT} \right)_{12} = \frac{S_1 - S_2}{V_1 - V_2} = \frac{L_{12}}{T(V_1 - V_2)}, \quad \Delta P = \rho g \Delta z. \quad (11)$$

$$\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + \dots, \quad 1 + x + x^2 + \dots + x^{N-1} = \frac{1 - x^N}{1 - x}. \quad (12)$$

Problems That Require Writing

Please write your answers to the following problems on extra blank sheets of paper, and make sure to write your name and the problem number at the top of each sheet. Unless otherwise stated, *you need to justify your answers to get full credit.*

1. **(6 points)** List six distinct criteria that a system has to satisfy to be in thermodynamic equilibrium. (No justification needed here.)
2. **(8 points)** Draw a representative phase diagram (with horizontal axis the temperature T and vertical axis the pressure P) that includes all the features that are typically observed in the phase diagram of a pure substance like carbon dioxide. Make sure to label clearly all features and regions of your plot. (No justification needed here.)
3. Over the semester, you learned about two models for which many thermodynamic details can be worked out analytically, the Einstein solid and an ideal gas.
 - (a) **(12 points)** In some detail (6 points worth each!), describe what *are* these two systems. Your discussion should state clearly what are the components of each system, and mention what assumptions, simplifications, and idealizations are made so that these models can be studied analytically.
 - (b) **(10 points)** With appropriate sketches showing axes and key graphical details clearly labeled, show how the normalized heat capacity $C_V/(Nk)$ varies with temperature for each of these two systems. Be particularly careful to draw the low- and high-temperature behaviors of $C_V/(Nk)$ correctly (e.g., if the curve approaches an axis, does it do so with a zero, finite, or infinite slope?). Where possible and as is appropriate, also indicate quantitative details on your plot, e.g., where room temperature ($T = 300 \text{ K}$) lies on your plot, and the approximate values on the axes where there is some physically relevant change in behavior.
 - (c) **(6 points)** Give an example for each model of a prediction that is *incorrect* experimentally, and explain briefly the reason for the incorrect prediction.

No derivations are needed for this problem, although some equations would be appropriate to mention.

4. **(6 points)** Estimate to one significant digit the height h in meters from which a penny at room temperature would have to fall so that the potential energy released, if delivered entirely to the penny, would raise the penny's temperature by one degree Kelvin.

Note: A penny has a mass of 2.5×10^{-3} kg and can be assumed to be pure zinc (it is actually 97.6% zinc coated with 2.4% copper). One mole of zinc has a mass of 6.5×10^{-2} kg.

5. **(6 points)** A planet is orbiting a binary star system (two stars that are orbiting each other). The planet is a distance d_1 from star 1 and a distance d_2 from star 2. The surface temperatures of the two stars are T_1 and T_2 and their radii are r_1 and r_2 , while the radius of the planet is R . Derive and write down an expression for the steady-state temperature T of the planet in terms of the data d_1 , d_2 , r_1 , r_2 , T_1 , T_2 , and R .

Note: For this problem, assume that the planet is a perfect blackbody emitter of uniform surface temperature, that the distances d_i change so slowly over time that they can be assumed to be constant, and that one star does not block any light from the other star.

6. Consider a system that is in thermodynamic equilibrium and that can exchange energy and particles with a reservoir whose constant temperature is T and whose constant chemical potential is μ .

- (a) **(6 points)** Derive and write down a formula for the heat capacity C of this system in terms of the grand partition function \mathcal{Z} .

Note: Only T , μ , the Boltzmann constant k , \mathcal{Z} , and some derivatives of \mathcal{Z} should appear in your answer.

- (b) **(4 points)** Describe briefly how to construct an experimentally feasible reservoir, and explain how (at least in principle) the values of μ and T can be varied independently for your reservoir. Such a reservoir would then allow you to study the properties of some system as a function of μ and T .

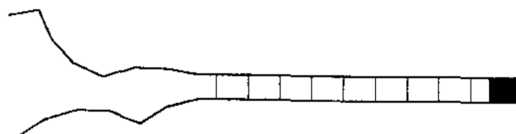
7. **(6 points)** A long vertical cylindrical tube contains a pure substance at temperature T in a gravitational field with constant gravitational acceleration g . Below a certain height z_0 in the column (as measured from the bottom of the column), the substance is found to be in a solid phase, while above that height, the substance is in a liquid phase. When the temperature of the entire column is decreased a little bit to a new value $T - \Delta T$, the solid-liquid interface is observed to move upwards a small amount to a new value $z_0 + \Delta z$. Neglecting the thermal expansion of the solid and liquid, derive and write down an expression for the mass density ρ_l of the liquid phase in terms of: the mass density ρ_s of the solid phase, the latent heat L per mass of melting, the gravitational acceleration g , the absolute temperature T , the temperature change ΔT , and the change in height Δz .

8. **(6 points)** Long after studying thermal physics, a student has forgotten a crucial detail of the Debye theory of a solid, and has replaced the Bose-Einstein distribution with a Fermi-Dirac distribution like this:

$$U = \frac{3\pi}{2} \int_0^{n_{\max}} \frac{hc_s}{2L} \frac{n^3}{e^{hc_s n/(2LkT)} + 1} dn, \quad n_{\max} = \left(\frac{6N}{\pi} \right)^{1/3}, \quad (13)$$

where the only error is a “+1” instead of a “-1” in the denominator of the integrand. By investigating the low- and high-temperature regime of this fermion version of Debye's theory, explain how this error leads to a prediction that is inconsistent with experiment.

9. A simple statistical physics model of the formation or unraveling of a long biomolecule like DNA as a function of temperature is a zipper



that has N links such that each link is closed with energy 0 or open with energy $\epsilon > 0$. The zipper can unzip only from one side (say from the left as shown above) and the n th link from the left can open only if all the links to the left of it ($1, 2, \dots, n-1$) are already open. The N th link on the right is always closed.

- (a) **(6 points)** By writing down an appropriate partition function for this zipper model, find an expression for the average number of open links \bar{n} .
- (b) **(4 points)** Show that \bar{n} becomes independent of N in the low-temperature limit and discuss the scientific meaning of this result.
10. **(8 points)** Experimentalists are able to create and study the properties of two-dimensional electron gases that float in a horizontal layer just above a liquid He surface. Consider such a gas of N electrons at absolute zero that occupies a square planar region of area $A = L^2$. What is the quantum degeneracy pressure P (force per unit length) in terms of the gas's energy density U/A ?
- Note: You can start with the following expression for the total energy U of the electrons

$$U = 2 \iint \epsilon(n_x, n_y) dn_x dn_y, \quad (14)$$

and change to polar coordinates for which $n = \sqrt{n_x^2 + n_y^2}$ and $dn_x dn_y = n dn d\theta$. Also observe that $\epsilon_F = \hbar^2 n_{\max}^2 / (8mL^2)$ and $N = 2 \times (\text{area of } 1/4 \text{ circle of radius } n_{\max} \text{ in } \mathbf{n} \text{ space})$.

Multiple Choice Questions (3 points each)

Circle the letter that best answers each of the following questions.

1. A system in thermodynamic equilibrium is quasistatically and adiabatically changed to some new equilibrium state. The thermodynamic quantity associated with this system that remains constant during this change is
- (a) energy U .
 (b) pressure P .
 (c) volume V .
 (d) entropy S .
 (e) chemical potential μ .

2. An object with constant heat capacity C_P and temperature T_1 is brought in contact with a thermal reservoir with constant temperature $T_2 \neq T_1$ until the object reaches equilibrium. The total change in entropy (of the object and reservoir) is then

- (a) $C_P \frac{T_2 - T_1}{T_2}$.
- (b) $C_P \left[\frac{T_2 - T_1}{T_1} - \frac{T_2 - T_1}{T_2} \right]$.
- (c) $C_P \ln \left(\frac{T_2}{T_1} \right)$.
- (d) $C_P \left[\ln \left(\frac{T_2}{T_1} \right) + 1 - \frac{T_1}{T_2} \right]$.
- (e) $C_P \left[\ln \left(\frac{T_2}{T_1} \right) - 1 + \frac{T_1}{T_2} \right]$.

3. If x is a small quantity so that $|x| \ll 1$, then the expression

$$\frac{1}{1 + ax + bx^2} \quad (15)$$

can be approximated to second-order accuracy by the expression $1 + cx + dx^2$ provided that the coefficient d is

- (a) b .
- (b) $-b$.
- (c) $-b - a^2$.
- (d) $-b + a^2$.
- (e) $-b + a^2 + 2ab$.

4. The multiplicity Ω of a one-dimensional ideal gas of N identical adsorbed particles of mass m moving freely about on a long carbon nanotube of length L is given by which of the following expressions:

- (a) $\frac{L}{N!} \frac{A_1(\sqrt{2mU})}{h}$.
- (b) $\frac{L^N}{N!} \frac{\left(A_1(\sqrt{2mU}) \right)^N}{h^N}$.
- (c) $\frac{L^N}{N!} \frac{A_{2N}(\sqrt{2mU})}{h^{2N}}$.
- (d) $\frac{L^N}{N!} \frac{A_N(\sqrt{2mU})}{h^N}$.

Note: The notation $A_d(r)$ denotes the surface area of a d -dimensional sphere of radius r .

5. A ferromagnet at absolute zero has all of its $N \gg 1$ spin-1/2 magnetic moments aligned in parallel (even in the absence of an external magnetic field). The ferromagnet is then heated until its temperature exceeds its Curie temperature, at which point the magnetization is zero. The change in entropy ΔS of the ferromagnet is then approximately given by

- (a) 0
- (b) $2Nk$

- (c) $2^N k$
- (d) $k \ln(2)$
- (e) $Nk \ln(2)$

6. The physicist Freeman Dyson has speculated about advanced civilizations that would harness the entire power of a star by surrounding it with an immense opaque shell that would trap all of the star's energy. Assume in the future that the human race disassembles the planet Venus (for which no one ever did find a good use) and uses its material to create a thin shell of thickness 100 meters that entirely surrounds the Sun, with a radius that is 1.1 times the radius of the Sun. After the Sun with its shell has reached a steady state, the difference ΔT_S between the the Sun's new surface temperature and previous surface temperature, and the difference ΔT_E between the Earth's new steady-state temperature and its previous temperature will satisfy

- (a) $\Delta T_S = 0$ and $\Delta T_E = 0$.
- (b) $\Delta T_S = 0$ and $\Delta T_E > 0$.
- (c) $\Delta T_S = 0$ and $\Delta T_E < 0$.
- (d) $\Delta T_S > 0$ and $\Delta T_E = 0$.
- (e) $\Delta T_S > 0$ and $\Delta T_E > 0$.
- (f) $\Delta T_S > 0$ and $\Delta T_E < 0$.

Note: For the purpose of the problem, assume that difference in radius of the shell and Sun can be ignored, and that the Sun and shell are perfect blackbody thermal emitters. Through its fusion reactions, the Sun emits the same constant total power, whether or not it is covered with a shell.