

Print your name clearly:_____

Signature:_____

“I agree to neither give nor receive aid during this quiz.”

Second Midterm Exam for Physics 176

Professor Greenside
Friday, March 27, 2009

This exam is closed book and will last the entire class period. Please note the following:

1. Only true-false and multiple choice questions should be answered on the exam itself.
2. All other questions should be answered on the extra blank pages. If you need extra pages during the exam, let me know.
3. Please write your name and the problem number at the top of each extra page.
4. Please write clearly. If I can not easily understand your answer, you will lose credit.
5. Please feel free to ask me questions during the exam if the wording of a problem is not clear.

The following formulas may be useful:

$$dU = TdS - PdV + \mu dN, \quad U = Q + W, \quad \bar{E} = -\frac{\partial \ln(Z)}{\partial \beta}, \quad C_V = \frac{1}{kT^2} \left(\overline{E^2} - (\bar{E})^2 \right). \quad (1)$$

$$F = U - TS, \quad dF = -SdT - PdV + \mu dN, \quad F = -kT \ln(Z). \quad (2)$$

$$\mathcal{D}(v) = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} v^2 e^{-mv^2/(2kT)}, \quad \bar{v} = \sqrt{\frac{8kT}{\pi m}}, \quad \overline{v^2} = \frac{3kT}{m}. \quad (3)$$

$$\int_0^\infty e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{2} \alpha^{-1/2}, \quad \int_0^\infty x e^{-\alpha x^2} dx = \frac{1}{2} \alpha^{-1}, \quad (4)$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad \int_0^\pi e^{a \cos(\theta)} \sin(\theta) d\theta = \frac{2}{a} \sinh(a). \quad (5)$$

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots, \quad \frac{1}{1+x} \approx 1 - x. \quad (6)$$

Problems That Require Writing

Please write your answers to the following problems on extra blank sheets of paper. Also make sure to write your name and the problem number at the top of each sheet. In this part of the exam, you need to justify all of your answers to get full credit.

1. **(8 points)** Consider a small system in contact with a large reservoir whose constant temperature is T , and assume that the system and reservoir consist of only one type of particle. Assume that the small system can exchange heat *and* exchange particles with the reservoir so neither the energy of the system nor the number of particles in the system is conserved. Using an argument similar to how the Boltzmann factor was derived and clearly stating your assumptions and approximations, **derive an expression for the probability for the small system to be in a state s that has energy $E(s)$ and that has $N(s)$ particles.** Your expression will look like an “exponential of something” over a quantity that looks like a partition function.
2. **(6 points)** Calculate the lowest-order approximation to the hyperbolic cotangent $\coth(x)$ for small x , i.e. deduce the values of the three numbers a , b and c for which

$$\coth(x) \approx \frac{a}{x} + bx + O(x^c), \quad |x| \ll 1. \quad (7)$$

3. Consider a paramagnet that consists of N distinguishable quantum magnetic dipoles $\boldsymbol{\mu}$. The paramagnet is immersed in an external magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ that has magnitude B and that is oriented along the positive z -axis. The paramagnet is also in thermodynamic equilibrium with a large reservoir whose constant temperature is T . Assume that the z -component of each dipole, μ_z , can take on the K different values m_1, m_2, \dots, m_K for $K > 1$ some integer. The K possible energy states $E(s)$ are given by the usual formula, $E(s) = -m_s B$.

- (a) **(6 points)** Show that the average magnetization \overline{M} of the paramagnet (M is the total magnetic moment along the z direction) can be calculated from the free energy F of the paramagnet via the equation

$$\overline{M} = -\frac{\partial F}{\partial B}. \quad (8)$$

- (b) **(2 points)** Explain how one could remember this formula by modifying the thermodynamic identity for F to include the variables M and B .
- (c) **(4 points)** Explain briefly whether the variables M and B are intensive or extensive quantities and why.

4. **Polarization of Classical Electric Dipoles in a Uniform Electric Field** The dielectric medium used in capacitors and the lipid membranes of biological cells consist of large molecules that act like electric dipoles whose average orientation changes in the presence of electric fields to a degree that depends on the ambient temperature. In this problem, you use partition functions to explore the competition between the electric field, which tends to order the dipoles, and temperature, which tends to disorder the dipoles.

You can think of an electric dipole \mathbf{p} as a neutral rigid rod with equal and opposite charges, Q and $-Q$ at each end, with the vector \mathbf{p} pointing from the negative to positive charge. In the presence of an external electric field \mathbf{E} , an electric dipole reaches a lowest energy state by aligning parallel to the electric field, exactly as a magnetic dipole $\boldsymbol{\mu}$ aligns parallel to an external magnetic field \mathbf{B} . At zero temperature, all electric dipoles in a medium would align parallel to an external uniform electric field, but at finite temperature, the disorder caused by fluctuations in the energies of the dipoles prevents full alignment. Unlike the magnetic dipoles that we have discussed so far, the electric dipoles associated with large molecules more typically act like classical, rather than quantum objects, with energies that vary continuously with changes in the orientations of the dipoles.

Assume that you have a collection of N classical non-interacting distinguishable electric dipoles \mathbf{p} in the presence of a uniform external electric field

$$\mathbf{E} = E \hat{\mathbf{z}}, \quad (9)$$

of magnitude E that points in the positive z direction. Assume also that the dipoles are in thermodynamic equilibrium with a large reservoir whose temperature is T .

A given electric dipole is described by a three-dimensional vector $\mathbf{p} = (p_x, p_y, p_z)$, and its energy U in the presence of the electric field Eq. (9) is given by

$$U = -\mathbf{p} \cdot \mathbf{E} = -p_z E. \quad (10)$$

The energy of a dipole depends on its orientation with respect to the electric field. We can describe this orientation with the two angles θ and ϕ of a spherical coordinate system that is located at the center of the dipole, where the z axis of the spherical coordinate system is chosen to be parallel to the electric field \mathbf{E} . The energy Eq. (10) can then be written as

$$U = -p_z E = -pE \cos(\theta), \quad (11)$$

where p is the magnitude of the vector \mathbf{p} . The partition function Z_1 for a single dipole will be a sum of Boltzmann factors over all possible energy states Eq. (11), which means a sum over all possible orientations (θ, ϕ) of the dipole such that $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. Because the angles are continuous variables, there is an effective degeneracy $d_{\theta, \phi} = d\Omega = \sin(\theta) d\theta d\phi$ since all dipoles in the infinitesimal solid angle $d\Omega$ centered on (θ, ϕ) have about the same amount of energy. The partition function Z_1 for a single dipole therefore has the form:

$$Z_1 = \sum_s e^{-\beta U(s)} \quad (12)$$

$$= \sum_{\theta} \sum_{\phi} d_{\theta, \phi} e^{-\beta U(\theta, \phi)} \quad (13)$$

$$= \iint d\Omega e^{\beta p E \cos(\theta)}. \quad (14)$$

Given this information, please answer the following questions.

- (a) **(2 points)** How is the partition function Z_N of all the electric dipoles related to the partition function Z_1 of a single dipole? You do not need to derive this, just give the relation.
- (b) **(3 points)** Evaluate the double integral in Eq. (14) to show that

$$Z_1 = 4\pi \frac{\sinh(\beta p E)}{\beta p E}. \quad (15)$$

- (c) **(4 points)** Using the analog of Eq. (8) for electric dipoles and electric fields (no derivation necessary), show that the average polarization \bar{P} is given by¹

$$\bar{P} = Np \left(\coth(\beta p E) - \frac{1}{\beta p E} \right). \quad (16)$$

- (d) **(2 points)** Determine a lowest-order large-temperature approximation to Eq. (16).
- (e) **(2 points)** Determine a lowest-order low-temperature approximation to Eq. (16).
- (f) **(4 points)** Using your low- and large-temperature approximations, draw a curve showing how, for fixed electric field strength E , the normalized polarization $\bar{P}/(Np)$ varies with the normalized temperature $kT/(pE)$ for $T \geq 0$, and discuss briefly whether your curve makes sense physically.

¹This answer simultaneously solves the problem of calculating the average length P of a three-dimensional polymer consisting of identical massless rigid rods of length p that are connected in such a way that one rod can rotate freely in three-dimensions with respect to the rods that it is attached to, and such that the two ends of the polymer are held with a constant tension E along the z -axis.

True or False Questions (2 points each)

For each of the following statements, please circle **T** or **F** to indicate whether a given statement is true or false respectively.

1. **T / F** If an engineer designs a device that will operate in equilibrium conditions such that combinations of pressure, temperature, and chemical potential might be held constant, then the appropriate thermodynamic potential (equivalent of the free energy) for the engineer to use during the design process is $U + PV - \mu N$.

2. **T / F**

$$\int_0^\infty x^7 e^{-x^2} dx = 3. \quad (17)$$

3. **T / F** If a single helium atom is added to a container of an ideal gas at temperature T that consists of much more massive xenon atoms, and if millions of measurements are made of the He atom's speed v over several days, then the experimentally measured speed distribution $\mathcal{D}_{\text{He}}(v)$ of the helium atom will be a Maxwell distribution.

4. **T / F** If a small system with three energy states $E_1 < E_2 < E_3$ is in thermodynamic equilibrium with a large reservoir whose temperature is T , then

$$\lim_{T \rightarrow \infty} \overline{E} = E_3. \quad (18)$$

5. **T / F** If a paramagnet A with negative temperature $T_A < 0$ is placed in thermal contact with a paramagnet B with temperature $T_B < T_A$, then heat will flow from A to B.

6. **T / F** If a small system has a finite range of energy states $E(s)$, so $a \leq E(s) \leq b$ for finite numbers a and b , then

$$\lim_{T \rightarrow \infty} C_V(T) = 0. \quad (19)$$

(Here $C_v(T)$ is the heat capacity for constant volume of the system when the system is in thermodynamic equilibrium with a large reservoir whose constant temperature is T .)

7. **T / F** For an equilibrium ideal gas of identical molecules of mass m , the average speed \bar{v} of the molecules and the average inverse speed $\overline{1/v}$ of the molecules satisfy

$$\bar{v} \times \overline{1/v} < 1. \quad (20)$$