

Print your name clearly:_____

Signature:_____

“I agree to neither give nor receive aid during this exam.”

Final Exam for Physics/ECE 176

Professor Greenside
Wednesday, May 4, 2011

Please read the following before starting the test:

1. This exam is closed book and will last the entire exam period.
2. No calculators or other electronic devices are allowed.
3. Look over the entire exam and get a sense of its length, what kinds of questions are being asked, and which questions are worth the most points.
4. Answer the true-false and multiple choice questions on the exam itself, answer all other questions on extra blank pages. If you need extra pages during the exam, let me know.
5. Please write your name and the problem number at the top of each extra page.
6. Unless otherwise stated, you must justify any written answer with enough details for me to understand your reasoning.
7. Please write clearly: if I can not *easily* understand your answer, you will lose credit.
8. If you are not sure about the wording of a problem, please ask me during the exam.

Problems That Require Writing

1. **(8 points)** A germ can be approximated as a rigid sphere of radius about a micron (10^{-6} m) that is filled with water (density $\approx 10^3$ kg/m³). To the nearest power of ten and in units of microns per second (i.e., body lengths per second), estimate the magnitude of the germ's speed that arises from the germ being in equilibrium with pond water at room temperature.
2. **(12 points)** During the semester, you learned two paradoxical facts: that “it is not possible to boil water with boiling water” but “it is possible to boil water with ice”. Explain briefly how to demonstrate these statements experimentally, and also discuss briefly the physics that explains these facts.
3. **(8 points)** What is the generalization of the Gibbs factor $e^{-\beta(E_s - \mu N_s)}$ for the case of a small system like a porous balloon that can exchange energy E_s , particles N_s , and volume V_s with some reservoir?

4. (a) **(5 points)** Describe precisely what is meant by “Einstein’s model of a solid”.
 (b) **(20 points)** Assume that an Einstein solid that consists of N identical atoms is in equilibrium with a reservoir whose temperature has the constant value T . Derive an expression for the heat capacity $C(T)$ of the Einstein solid.
 Next, plot $C/(Nk)$ qualitatively versus kT/ϵ , where ϵ is the constant spacing between energy levels, and give some numerical values on the horizontal and vertical scales to indicate the scales involved. Also discuss briefly how your plot relates to the equipartition theorem and to the third law of thermodynamics.
 (c) **(10 points)** Motivated by Einstein’s model, the Dutch scientist Peter Debye obtained a more accurate description of the heat capacity $C(T)$ of a solid by using more realistic assumptions than those of Einstein. His theory led to the following result for the thermal energy U of a solid consisting of N atoms in a volume V at temperature T :

$$U = \frac{9NkT^4}{T_D^3} \int_0^{T_D/T} \frac{x^3}{e^x - 1} dx, \quad (1)$$

where the so-called Debye temperature T_D is given by

$$T_D = \frac{hc_s}{2k} \left(\frac{6N}{\pi V} \right)^{1/3}, \quad (2)$$

where c_s is the speed of sound waves in the solid, and is assumed to be a constant.

Given Eqs. (1) and (2), make two experimentally testable predictions of Debye’s theory by exploring the low-temperature behavior and high-temperature behavior of the heat capacity $C(T)$. Hint: it is easier to find first the leading order behaviors of the energy $U(T)$ and then differentiate those to get the leading order behaviors of $C(T)$.

5. (a) **(10 points)** Let t_{mountain} be the time to cook noodles in boiling water until they are soft and ready to eat at a mountain resort that is three kilometers (about 10,000 ft) above sea level, and let $t_{\text{sea-level}}$ be the time to cook the same noodles until ready to eat at sea level. By referring to a temperature-pressure phase diagram of water (a crude sketch will suffice), explain qualitatively why $t_{\text{mountain}} > t_{\text{sea-level}}$.
 Note: the time for chemical reactions to run to completion depends sensitively on the temperature of the medium containing the reactions.

- (b) **(10 points)** Now answer this question quantitatively. Assume that steam can be treated as an ideal gas satisfying $PV = NkT$, that the latent heat $L > 0$ for a kilogram of water to vaporize into steam can be treated as constant, and that the volume of a kilogram of water is negligible compared to the volume of a kilogram of steam. Use the Clausius-Clapeyron equation to derive a mathematical expression for the phase boundary $P(T)$ that separates water and steam.

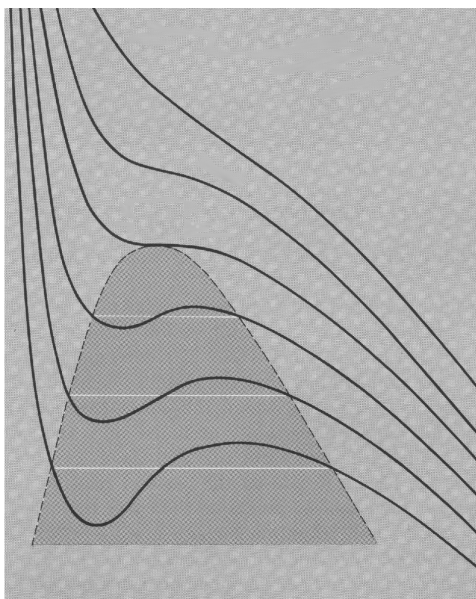
Now recall that, for an equilibrium atmosphere with temperature T_0 , the pressure P decreases with height z above sea level as $P = P_0 e^{-mgz/(kT_0)}$. Combine this result with your expression for the phase boundary $P(T)$ to derive how the temperature $T(z)$ for water to boil varies with height.

6. (a) **(5 points)** The van der Waals equation

$$\left(P + a \frac{N^2}{V^2}\right)(V - Nb) = NkT, \quad (3)$$

contains two corrections to the ideal gas law. Describe briefly and qualitatively the motivation for making these two corrections.

- (b) **(10 points)** Discuss how the following figure



is related to the claim that the van der Waals equation is able to describe the phase transition of a liquid to a gas. Your discussion should address all key details of the plot: what do the axes represent, what are the successive continuous curves, what are the horizontal white lines, what is the meaning of the darker shaded region, which parts of the plot corresponds to the liquid and gas phase, and which part of the plot corresponds to the critical point. Also explain briefly the conceptual steps that were carried out to generate this plot.

7. An important practical problem in electrical engineering and material science is the manufacture of ultrapure crystals, say of silicon, for making integrated circuits. A difficulty in creating such ultrapure crystals is that thermodynamic equilibrium allows some atoms to move out of their normal lattice positions to so-called interstitial locations between atoms. Atoms in “wrong” locations or absent from their normal locations are called defects and defects can substantially influence, often negatively, the electrical and mechanical properties of a crystal.

Explore this situation by considering an idealization of a crystal that has N points periodically spread out in space and N interstitial positions between the lattice points where an atom can reside. Let $\epsilon > 0$ be the energy needed to remove an atom from a lattice point and move it to any interstitial location, and let n be the number of atoms occupying interstitial sites.

- (a) **(5 points)** What is the internal energy U of the crystal when there are n defects, in terms of the internal energy U_0 when there are no defects?
- (b) **(10 points)** Use a combinatoric argument to find an approximate expression for the entropy S of the solid with n defects with the assumption $N, n \gg 1$.
Hint: think about how many ways you can choose n atoms out of N lattice sites to move to interstitial locations, and think also of how many ways you can choose n interstitial sites out of N possible such sites to receive the n atoms.
- (c) **(10 points)** Derive an equation that describes how, in thermal equilibrium, the number of defects n depends on the temperature T of the crystal as well as on the parameters N and ϵ . Discuss whether your solution makes physical sense.
Hint: assuming that the volume and temperature are fixed, you will want to minimize some thermodynamic potential as a function of n .

8. Work out a theory of a *two*-dimensional equilibrium photon gas by answering (with appropriate justifications) the following questions:

- (a) **(3 points)** Is the chemical potential μ of a 2D photon gas still equal to zero?
- (b) **(6 points)** What is the Planck spectrum $u_{2D}(\epsilon)$ of the two-dimensional photon gas, as a function of photon energy ϵ ?
For the 3D case,

$$u_{3D}(\epsilon) = \frac{8\pi}{(hc)^3} \frac{\epsilon^3}{e^{\beta\epsilon} - 1}. \quad (4)$$

- (c) **(8 points)** From the properties of a 2D equilibrium photon gas, calculate the adiabatic exponent γ that determines how the pressure P and two-dimensional “volume” A (area) are related, $PA^\gamma = \text{constant}$, during an adiabatic (constant-entropy) process of the gas.
For a 3D photon gas, one has $PV^\gamma = \text{constant}$ with $\gamma = (f + 2)/f = 4/3$. This implies naively that a three-dimensional photon corresponds to $f = 2/(\gamma - 1) = 6$ degrees of freedom.
- (d) **(8 points)** What is the 2D version of Stefan’s radiation law: how does the power radiated per unit length depends on temperature T ?
Hint: use kinetics to calculate the energy of photons that leave through a small opening of length L in a square box containing the 2D photon gas. For 3D, the power emitted per unit area is proportional to T^4 .

Note that two-dimensional photons have only one possible polarization. Also, at some point you will need to use polar coordinates and the fact that $dn_x dn_y = n dn d\theta$.

Multiple Choice (3 points each)

For each of the following statements, please circle the letter that best answers the question.

1. If a Carnot cycle of an ideal gas is drawn instead on a temperature-entropy plane for which the horizontal axis is temperature $T \geq 0$ and the vertical axis is entropy $S \geq 0$, then the physical meaning of the area enclosed by the cycle on the $T - S$ plane is
 - (a) the total work done on the gas.
 - (b) the total heat transferred to the gas.
 - (c) the total entropy change associated with the two isothermal processes.
 - (d) the total entropy change of the gas plus environment.

2. If the size L of a macroscopic cubic nonequilibrium system is increased by a factor of 2 and the largest temperature difference ΔT across the system is decreased by a factor of $1/2$ (without changing the physical properties of the system), then the relaxation time τ will
 - (a) remain unchanged.
 - (b) increase by a factor of 2.
 - (c) decrease by a factor of $1/2$.
 - (d) increase by a factor of 4.
 - (e) decrease by a factor of $1/4$.

3. Which of the following systems has a negative heat capacity?
 - (a) a gas consisting of spin- $3/2$ magnetic dipoles in a uniform external magnetic field.
 - (b) rubber.
 - (c) a solution of hemoglobin molecules.
 - (d) the Sun.
 - (e) no system since the heat capacity must be a non-negative quantity.

4. A thermally isolated spin- $1/2$ paramagnet is in thermodynamic equilibrium in the presence of an external uniform magnetic field of strength B . If the magnitude B is doubled in value, then the heat capacity curve $C(T)$ for $T > 0$ of the paramagnet will
 - (a) remain unchanged.
 - (b) will increase for each temperature value.
 - (c) will decrease for each temperature value.
 - (d) will increase for some temperature values and decrease for other temperature values.

True or False Questions (2 points each)

For each of the following statements, please circle **T** or **F** to indicate whether a given statement is true or false respectively.

1. **T / F** The entropy of a mole of molecules in liquid form is always greater than the entropy of a mole of the same molecules in solid form.

2. **T / F** If a fermion and boson bind together to form a hybrid particle, that particle will act like a fermion.

3. **T / F** The degeneracy pressure of an electron gas in a metal is mainly due to the mutual electrical repulsion of the like-charged electrons.
4. **T / F** If the temperature of a block of copper metal is doubled from 200 K to 400 K, then the entropy S of the electron gas in the metal will increase by a factor of 2.
5. **T / F** The thermal average $\langle X \rangle$ of some macroscopic quantity X does not depend on temperature.
6. **T / F** The denser of two thermodynamic phases of the same substance always becomes the one observed experimentally as the pressure on the substance is steadily increased.
7. **T / F** Because of the change in gravitational potential energy mgz , the average speed \bar{v} of a gas molecule varies with height z for a gas in thermodynamical equilibrium with temperature T .
8. **T / F** If at room temperature the rotational energy scale ϵ_{rot} for diatomic hydrogen were suddenly made twice as small, the pressure of the gas would change.
9. **T / F** If a certain system has N particles, then the expression $NkT[(1 + \ln(cP/N))]$, where P is the pressure of the system and c is some positive constant, would be a physically acceptable expression for the entropy S of the system.
10. **T / F** It is possible to find a thermodynamic potential $X(T, P, V)$ that is a function of the temperature T , pressure P , and volume V .
11. **T / F** The Chandrasekhar limit for the maximum mass possible for a white dwarf star would be smaller if electrons were bosons instead of fermions.
12. **T / F** If an ideal gas of identical molecules in some container is such that the molecular velocities are isotropic and the molecules all have the same speed v with a value determined by the equipartition theorem, $(1/2)mv^2 = (3/2)kT$, then the gas is *not* in thermodynamic equilibrium.
13. **T / F** If $\mathcal{D}_{2D}(v)$ is the Maxwell speed distribution for an ideal equilibrium **two**-dimensional gas consisting of identical molecules moving in a two-dimensional square box of area A , then the speed distribution $\mathcal{D}_{\text{eff}}(v)$ of particles that effuse through a tiny hole in the side of box is given by $\mathcal{D}_{\text{eff}}(v) = (v/v_{\text{avg}})\mathcal{D}_{2D}(v)$, where v_{avg} is the average speed of the two-dimensional molecules.
14. **T / F** For the zipper model of a DNA molecule that you studied earlier this semester, the partition function Z_{total} for the entire molecule of N links is given by $Z_N = Z_1^N$ where $Z_1 = 1 + e^{-\beta\epsilon}$ is the partition function for a single link. (Recall that an existing link has zero energy while a broken link has energy $\epsilon > 0$.)

The following data and formulas may be useful:

$$PV = NkT, \quad \Delta U = Q + W, \quad U = Nf(kT/2), \quad v_{\text{rms}} = \sqrt{3kT/m}, \quad (5)$$

$$PV^\gamma = \text{const}, \quad VT^{f/2} = \text{const}, \quad \gamma = \frac{f+2}{f}. \quad (6)$$

$$C = \frac{Q}{\Delta T}, \quad L = \frac{Q}{m}, \quad C_V = \left(\frac{\partial U}{\partial T} \right)_V, \quad C_P = \left(\frac{\partial U}{\partial T} \right)_P + P \left(\frac{\partial V}{\partial T} \right)_P. \quad (7)$$

$$S = k \ln \Omega, \quad S = \frac{Q}{T}, \quad \Delta S = \int_{T_1}^{T_2} \frac{C(T)}{T} dT. \quad (8)$$

$$\overline{E} = -\frac{\partial \ln(Z)}{\partial \beta}, \quad C_V = \frac{1}{kT^2} \left(\overline{E^2} - (\overline{E})^2 \right), \quad \overline{N} = -\frac{\partial \ln(Z)}{\partial x}, \quad x = \beta(\epsilon - \mu). \quad (9)$$

$$\mathcal{D}(v) = \left(\frac{m}{2\pi kT} \right)^{3/2} 4\pi v^2 e^{-\beta m v^2/2}, \quad 1 = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-\beta m v^2/2}. \quad (10)$$

$$v^2 = v_x^2 + v_y^2 + v_z^2, \quad dv_x dv_y dv_z = v^2 \sin(\theta) dv d\theta d\phi, \quad dv_x dv_y = v dv d\theta. \quad (11)$$

$$F = U - TS, \quad dF = -S dT - P dV + \mu dN, \quad Z = \sum_s e^{-\beta U(s)}, \quad F = -kT \ln(Z). \quad (12)$$

$$G = U - TS + PV, \quad dG = -S dT + V dP + \mu dN, \quad G = N\mu. \quad (13)$$

$$\left(\frac{\partial G}{\partial P} \right)_{T,N} = V, \quad \left(\frac{\partial G}{\partial T} \right)_{P,N} = -S, \quad \left(\frac{dP}{dT} \right)_{12} = \frac{S_1 - S_2}{V_1 - V_2} = \frac{L_{12}}{T(V_1 - V_2)}. \quad (14)$$

$$\Phi = U - T S - \mu N, \quad d\Phi = -S dT - P dV - N d\mu, \quad \mathcal{Z} = \sum_s e^{-\beta[U(s) - \mu N(s)]}, \quad \Phi = -kT \ln(\mathcal{Z}). \quad (15)$$

$$U = \alpha \sum_{n_x} \sum_{n_y} \sum_{n_z} \epsilon(\mathbf{n}) \bar{n}[\epsilon(\mathbf{n})], \quad \bar{n}_{FD} = \frac{1}{e^{\beta(\epsilon - \mu)} + 1}, \quad \bar{n}_{BE} = \frac{1}{e^{\beta(\epsilon - \mu)} - 1}. \quad (16)$$

$$p_n = \frac{h}{\lambda_n} = \frac{hn}{2L}, \quad \epsilon_{\text{electron}} = \frac{p^2}{2m} = \frac{h^2 n^2}{8mL^2} = \frac{h^2}{8mL^2} (n_x^2 + n_y^2 + n_z^2), \quad \epsilon_{\text{photon}} = pc = \frac{hcn}{2L}. \quad (17)$$

$$\epsilon_F = \frac{h^2}{8m} \left(\frac{3N}{\pi V} \right)^{2/3}, \quad U = \frac{3}{5} N \epsilon_F, \quad P = \frac{2}{3} \frac{U}{V}. \quad (18)$$

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}, \quad \ln(n!) \approx n \ln n - n, \quad \ln(1+x) \approx x. \quad (19)$$

$$\frac{dx}{dy} = c \frac{x}{y^\beta} \Rightarrow x = \tilde{c} \exp \left[\frac{c}{1-\beta} y^{1-\beta} \right], \quad \beta \neq 1. \quad (20)$$

$$1 + x + x^2 + \dots = \frac{1}{1-x} \quad \text{for } |x| < 1, \quad e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \dots, \quad (21)$$

Score:

Grade:

1. Problem 1:
2. Problem 2:
3. Problem 3:
4. Problem 4:
5. Problem 5:
6. Problem 6:
7. Problem 7:
8. Problem 8:
9. Multiple choice:
10. True-false:

Total points lost: