Answers to Physics 176 One-Minute Questionnaires
Lecture date: February 15, 2011

What is the historical reason that there is an offset of 1 in the definition of the Gamma function, $\Gamma(x+1)$?

I don't know. If you somehow find out, please let me know.

By the way, if you like mathematics enough to enjoy reading mathematics books, I would recommend the book *Gamma: Exploring Euler's Constant* by Julian Havil. It is one of the few books that I know of that gives an exciting sense of the many interesting ideas and discoveries that Euler and his contemporaries explored. The book includes a discussion of the Gamma function and also several other famous functions like the zeta function that relates to the theory of prime numbers. It is not technical, nearly everything can be understood with a freshman level of mathematics (some calculus and a little number theory).

What is the error in Stirling's approximation?

The error is often written as an infinite series involving powers of the small quantity $1/n$ where $n \gg 1$ is the integer for which you are trying to estimate $n!$.

Problem B.13 on page 391 of Schroeder gives one way to calculate the errors associated with Stirling's approximation but the recommended heavy-duty way to calculate the error systematically is a valuable result of classical applied mathematics called the Euler-Maclaurin formula. An explicit expression for the error is given in the Wikipedia article "Euler-Maclaurin formula".)

If you ever take a course in computational or numerical mathematics, you will see frequent use of the Euler-Maclaurin formula since it provides a deep understanding of how well one can approximate the value of an integral numerically by a finite sum of values of the integrand.

Can you point me in the direction of a more formal discussion of the fact that $f(x)^N \approx$ Gaussian for large N?

The argument I gave in lecture gave all the key insights of why this result is true, not sure a more rigorous argument will give you a better understanding. My enhanced version of Schroeder's problem 2.1 on page 66 should also convince you visually that the approximation works quite well, even for N a small number (of order 100 or less).

That being said, I have never seen a rigorous formal proof of this result. The result is regarded (at least by physicists) as "trivial" once you see that it depends only on taking a log and doing two Taylor series approximations and I have never been motivated to chase down the details required for a careful proof.

What other numbers fall into the fine-tuning paradox?

These are described in Martin Rees's short non-technical book *Just Six Numbers: The Deep Forces that Shape The Universe.* Other numbers besides the ratio of electrical to gravitational forces are

1. the fractional amount of energy ϵ released when four hydrogen atoms fuse to produce helium and energy (relative to the rest mass of the four hydrogen atoms);
2. the geometric curvature Ω of the universe, which summarizes in a single number the total mass of the universe which in turn determines the future of the universe, for example whether it will expand forever or start contracting and collapse.
3. A cosmological constant λ that appears in the Einstein gravitational equations that is the cause of the accelerated expansion of the universe.
4. Then number of spatial dimensions, $D = 3$. Various laws of physics, and their ability to lead to interesting structure, change if you switch to 2 or 4 dimensions.

I have forgotten the sixth constant.

As usual,Wikipedia has some interesting things to say under the title "Fine-tuned Universe" but also, as usual, take what Wikipedia says with a grain of salt.

Why did you say Gaussians appear so much in nature and physics?

One reason is a basic result of probability theory called the "central limit theorem", that the average of a lot of independent random variables satisfies a Gaussian distribution. Since nearly all measurements on macroscopic systems involve a sum of many variables (a macroscopic sensor like a thermometer or pressure gauge is in contact with many fluctuating parts of a substance and so responds to their mean), Gaussian distributions occur in most experimental measurements.

Another reason is what we discussed earlier, that many phenomenon regarding the approach to equilibrium can be understood in terms of transfer of information by a random walk. The final location of the random walk is the sum of many independent random steps and so satisfies the central limit theorem and again one finds Gaussian. See Problem 2.25 on page 67 of Schroeder which gives you a chance to explore this connection. The solution of diffusion equations (either for heat or particles) involves a superposition of Gaussians, basically because diffusion equations are derived from an underlying random walk.

A third reason has to do with the uncertainty principle of quantum mechanics, that the Gaussian curve (wave packet if you like) is the one that achieves simultaneously the smallest width in real space and the smallest width in momentum space. This result makes Gaussians valuable for many practical applications, e.g., generating light pulses or for filtering signals.

A fourth reason is that Gaussians occur in a variety of quantum mechanical problems, for example they show up in the wave function of the quantum harmonic oscillator and the harmonic oscillator.

How do we know that the universe is 10^{14} years old? Couldn't it just be that objects past the light cone are too redshifted to be detected, or that space is expanding too fast for the light to ever reach us?

There are multiple independent observations which suggest that nothing in the universe is older than about 14 billion years. One is our excellent understanding of the ages of stars, and we don't see any stars that are older than about the age of the universe, no matter how far we look out into space. Another estimate of age comes from the decay rates of various isotopes that are found on the Earth, on the Moon, and in various meteorites that land on the Earth, and these all are consistent with the elements first being formed about billion of years ago but not longer. A third estimate comes from the temperature 2.7 K of the microwave cosmic radiation filling space, and subtle details of the deviation of that photon gas from thermal equilibrium. A fourth estimate comes from extrapolating the expansion of the universe backwards in time, we now have highly precise data that all point to a Big Bang of about 14 billion years ago.

Although the evidence is convincing (at least to me) that nothing is older than about 14 billion years, you are justified in pointing out that this does not mean that the universe is only 14 billion light years in size. General relativity allows space to expand more rapidly than the speed of light and so it could be that our 14 billion year old universe is much bigger than we can currently perceive. Perhaps the gravitational wave telescopes, as they turn and collect data, will provide some new insights about this question.

Is there any cause for the expansion of the universe?

There is currently no known mechanism to explain the accelerating expansion, and this remains a high priority scientific problem for physicists and astronomic to try to understand.

I have not yet looked at the homework and I am familiar with Gamma functions, but how do they related to our discussion of thermal physics?

We will derive and discuss this Thursday Eq. (2.40) on page 71 of Schroeder, which gives the multiplicity of an ideal gas. The formula requires defining factorials for half-integer values which in turn requires knowing about the Gamma function. But this will be our only direct need for using $\Gamma(x)$ this semester. I will use it again briefly when we get to Section 6.4 of Schroeder since various moments of the equilibrium speed distribution of a gas are most easily computed in terms of the Gamma function.

Do you think Watson or the humans will win in Jeopardy?

Let me ask Watson that question.