Answers to Physics 176 One-Minute Questionnaires Lecture date: February 10, 2011

Are Einstein solids related to Bose-Einstein condensates?

No, they are not. Einstein's model of a solid addresses the vibrational properties of a crystal, for example how vibrations of atoms in crystals contribute to the heat capacity. A Bose-Einstein condensate is a quantum state of matter that occurs only at extremely low temperatures. The condensate arises when a collection of identical massive bosons (a certain class of fundamental particles that have integer spin, such as a gas of rubidium atoms) can be described by a single "coherent" macroscopic wave function. We will discuss briefly the properties of Bose-Einstein condensates and why they occur later this semester.

What is the purpose and motivation behind calculating Ω_{total} ?

The goal of Section 2.3 of Schroeder and of the related discussion in class was to explain why the multiplicity of some macroscopic object spontaneously increases toward a maximum value, and why the transfer of energy (heat) between subsystems is an irreversible probabilistic phenomenon.

The first step in the discussion was to calculate the multiplicity Ω_{total} of a thermally isolated Einstein solid that consists of two interacting macroscopic subsystems A and B. Our discussion led to several insights. One was to label a macrostate of the total solid by the amount of energy q_A in subsystem A, which then determines the amount of energy $q_B = q - q_A$ in subsystem B since energy is conserved in the isolated total system.

A second important insight was that, generally, two subsystems interact weakly with one another (they affect each other only for atoms near the surface where they contact one another). This implies that the behavior of subsystem A is statistically uncorrelated with the behavior of subsystem B (at least on times short compared to a relaxation time) which in turn implies that the total multiplicity is approximately equal to the product of the multiplicities of the subsystems:

$$\Omega_{\text{total}} \approx \Omega_A \times \Omega_B. \tag{1}$$

This insight allows us to calculate Ω_{total} for any specified amount of energy q_A in subsystem A which led to a third insight as discussed in Section 2.3 of Schroeder: the total multiplicity $\Omega_{\text{total}}(q_A)$ generally has a large narrow peak as a function of q_A (especially if number of oscillators in each subsystem is large). This means that some macrostates have many more accessible microstates than other macrostates and so, if all accessible microstates are equally likely, certain macrostates are much more likely to be observed than others. This explains why the multiplicity of a closed system will "spontaneously" increase toward a maximum value, corresponding to the macrostate with the most accessible microstates.

In this Tuesday's lecture, when I discuss pages 64-66 and 72-74, we will refine this insight by showing mathematically that the peak of $\Omega_{\text{total}}(q_A)$ becomes fantastically narrow (of order $1/\sqrt{N} \approx 10^{-10}$ for $N \approx N_A$) for macroscopic objects with of order Avogadro's number of molecules. This narrow peak implies that only one value of q_A will ever be observed experimentally when dealing with an isolated equilibrium system.

What if the area/volume ratio is bigger, for example one pancake lying on top of another? Of is the higher dimension of the volume what matters?

Two thin pancakes lying adjacent along their areas would be an example for which the interaction energy could be comparable to the bulk thermal energies. In this case, the total multiplicity would not be the product of the multiplicities of each pancake, $\Omega_{\text{total}} \neq \Omega_A \times \Omega_B$, and we would not be able to define an entropy via $S = k \ln \Omega_{\text{total}}$ since this quantity would not be additive over subsystems. The relaxation time for heat transfer between pancakes of different temperatures could also be so short that it might not be meaningful to label a macrostate of the two-pancake system by the energy q_A in one pancake.

What drives the assumption that all microstates are equally likely?

It is the simplest hypothesis to make in the absence of any knowledge that would explain why one microstate might be favored over another microstate.

Most gases, liquids, and solids are statistically homogeneous, the physical properties of one macroscopic region (say a cubic millimeter) are the same for all other regions except extremely close to the surface of a substance. So one would expect each macroscopic region to have similar microstates. Further, each macroscopic subsystem varies in a complex way in time (nuclei vibrating, electrons moving through the crystal and scattering off other electrons and nuclei) and so you can think of a given subsystem of atoms as being kicked by all the surrounding regions, which you might guess would force the subsystem to switch to different microstates over time. In the end, it is a simple and reasonable hypothesis that leads to excellent agreement with experiment. It is just bad luck for theoretical physicists that it is a particularly difficult assumption to derive for some model of a macroscopic system.

What is the relationship between the highly probable macrostate and some kind of macroscopic equilibrium? Will an equilibrium system always be in the most probable macrostate?

The most probable macrostate of an isolated system is the equilibrium state with one qualification: the system has to have enough components N that the width of the peak in the multiplicity curve $\Omega(N_1, U_1, V_1)$, as a function of quantities that can vary between subsystems, is extremely narrow (see Figures 2.6 and 2.7 in Schroeder). Otherwise, you could have situations like Fig. 2.4 and 2.5 in Schroeder's book in which there is a range of macrostates that could be observed.

Your question is not just academic: an important frontier of science and technology is nanoscience, the properties of natural and artificial systems that are 10-10,000 atoms in diameter. These systems can still be studied and understood in terms of thermal physics but now N is rather small and there is not a unique equilibrium state that will be observed. We won't study such small systems in this course.

You should also be aware that most equilibrium systems of interest are not isolated ones but are in contact with other equilibrium systems such that it is the pressure P and temperature T, not the volume and total energy, that are kept fixed during some experiment. In this case, it is not the entropy of the system that reaches a maximum for equilibrium to occur but some other quantity like -G, the negative of the so-called free energy G = U - TS + PV. (We will discuss this later this semester.) For these cases of a system that is not isolated, the equilibrium state no longer corresponds to a maximum in the multiplicity.

The conclusion is that for isolated systems, the state with the maximum multiplicity will correspond to equilibrium but there are non-isolated equilibrium systems, ones with constant temparture, pressure, or chemical potential, for which this is not true. For liquids and gases that do not behave like Einstein solids, how do we get to the same probabilistic result that it is highly probable to be in the macrostate where the energy is split proportionally between the subsystems?

We will answer your question this Tuesday and Thursday, when we discuss pages 64-66 and 72-73 of Schroeder, how the multiplicity $\Omega(U_A)$ is a very sharply peaked function for macroscopic Einstein solids and for gases. The argument for liquids turns out to be the same for gases since the key difference between a solid is that the molecules in liquids and gases have kinetic energy and can move about.

Is there some very tiny probability of entropy decreasing in the universe over the next millisecond?

We don't know enough about the entire universe to answer this question, e.g., we don't know if the universe forms an isolated system and we don't know the properties of the "dark energy" that is causing the universe to expand at an accelerating rate.

But if you focus your attention on any particular region in the universe, there is a nonzero tiny chance for the entropy to spontaneously decrease. The smaller the system, the more likely this possibility, dS/dt > 0 only holds accurately for macroscopic systems.

Kind of a metaphysical question—does math come from nature or from our heads?

This is a difficult, fascinating, and unsettled question that has been discussed by philosophers and scientists over many centuries, and increasingly by psychologists and neurobiologists in the 21st century.

Experiments by the psychologist Elizabeth Spielke and others (including Elizabeth Brannon at Duke) have shown the unexpected result that human babies and animals of many species have an innate ability to count. For example, researchers have invented ingenious experiments in which they were able to show that many animals can distinguish numbers up to about five in an abstract way (abstract in that animals can tell that four is different than five in a way that doesn't depend on the choice of objects). So there is strong evidence that evolution has led an innate ability to count (at least up to small integers), which is a kind of mathematics that you could argue comes from nature. You are really asking two questions. One is how is it that humans are able to do abstract mathematics of any kind, say use variables like x, discover geometric theorems, manipulate algebraic expressions, deal with concepts like infinity and topology? Some scientists have guessed that this is a side effect of what all brains have been found to do to help an animal survive in the world, which is to construct internal models of how an animal's body moves in response to the world. Thus experiments show that animals are skilled at estimating the path of a thrown object and calculating an escape plan, all without knowing math or physics. It is not to hard to guess that this capability could be taken off line, e.g. an animal could spend time thinking or dreaming about possibly dangerous situations and trying to figure out how to escape them, and then at some point this becomes some kind of abstract capability that could lead to mathematical ability.

A second question, the one more relevant for scientists and engineers, is: why is it that the mathematics that humans have learned to do turns out to be so incredibly useful for understanding the world and the universe? For example, why is it that, except in the limits of very high energies (the Big Bang and particle physics) and large masses (black holes), all of nature can be quantitatively and accurately described and predicted by about three pages of known equations? (These would be equations like Einstein's theory of general relativity, the Schrodinger equation of quantum mechanics, the Maxwell equations of electrodynamics, the Navier-Stokes equations, the relativistic extension of Newton's equations for motion in the presence of electrical and gravitational fields, and perhaps quantum chromodynamics.) Why are the laws of nature so simple and concise when expressed in mathematical form?

The answer is not known and the question remains intriguing. You might enjoy reading a classic essay by the Physics Nobel prize winner Eugene Wigner, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences", which you can find on the Internet, for example here:

http://www.dartmouth.edu/~matc/MathDrama/reading/Wigner.html

How are very large numbers conceptualized?

This is a tricky question: what do you mean by "conceptualize"?

Many scientists and mathematicians are able to gain intuition about abstract entities like large numbers, four or more spatial dimensions, the hierarchy of infinities (\aleph_0 , \aleph_1 , etc), the quantum mechanical wave function Ψ of a physical system, and so on by solving various problems and manipulating these objects in various contexts. Such symbolic manipulations and problem solving make one at least familiar with such concepts and, in some cases, one is able to conceptualize what is going on by manipulating the related mathematics. For example, some mathematicians are able to manipulate mathematics so quickly and adeptly that they can effectively think about geometric shapes in four or higher spatial dimensions. (A problem from the 2008 Putnam exam: what is the largest radius of a circle contained in a 4-dimensional hypercube of side length 1?)

I am not sure that anyone really is able to appreciate numbers like a googol or googolplex or especially Graham's number, they are immense in a real physical sense, e.g., a googol already exceeds the number of subatomic particles in the entire known universe. One can carry out various symbolic manipulations, one can order by size various large numbers and so one can become familiar with many large numbers and their properties (like multiplying a very large number by a large number doesn't change the very large number). But I am not sure we can grasp such immense numbers in any tangible way.

What was the practical use of Graham's number?

The Wikipedia article "Graham's number" gives a good summary of how the number arose, which was to determine that a certain very difficult problem in combinatorics had a solution, even if the solution could not be explicitly found and understood. I know of no practical use of this immense number except to remind people that the set of integers is truly infinite and that the numbers people and scientists use on a daily basis are an incredibly small and perhaps unrepresentative subset of all the possible integers.