Solutions

1. First, set up a coordinate system:

   (and forcing yourself to draw the physical system adds clarity for yourself & readers of your work)

   ![Diagram of a coordinate system with a slab and plane labeled with symbols]

   Note that the slab + plane are \( \infty \) in \( x+y \) directions (as I've chosen them)

   Think qualitatively first (not required for credit, but useful exercise.)

   What kind of fields do we expect outside of the slab-plane system? Symmetry tells us that the should only have \( \mathbb{E} \) components (imagine reflecting the system through the \( x-z \) or \( y-z \) planes - the physical system doesn't change, so the field should not). Symmetry also tells us that \( \mathbb{E}(x,y,z) = \mathbb{E}(z) \), that is, \( \mathbb{E} \) should not have a functional dependence on \( x+y \) (imagine translating the system in the \( x-y \) plane - again, same physical system \( \Rightarrow \) same fields). So we conclude that \( \mathbb{E} = \mathbb{E}(z) \), and we could now usefully apply Gauss's law, as you do in the next chapter.

   But that is unnecessary here, since we're given the \( \mathbb{E} \) field due to an \( \infty \) plane, which is information enough. Note that in the end, because of the \( \infty \) plane distribution, \( \mathbb{E} \) does not even depend on \( z \) - it's uniform on a given side of the plane.
Now, back to calculating.

Use superposition:

\[ \mathbf{E} = \mathbf{E}_{\text{plane}} + \mathbf{E}_{\text{slab}} \]

We're given

\[ |\mathbf{E}_{\text{plane}}| = 2\pi k \sigma \]

and we know it points away and \( \perp \) to plane.

We'll use this idea to calculate \( \mathbf{E}_{\text{slab}} \) by slicing the slab into infinitesimally thin planes:

\[ \mathbf{E}_{\text{slab}} = \int \mathbf{dE}_{\text{slice}} \]

The fields due to one slice look like

\[ \mathbf{dE}_{\text{slice}} = 2\pi k d\sigma \]

This might often be written as just \( \sigma \), but I think this will be clearer here.
1. (cont'd)

What is \( \sigma \)?

If a combination of dimensional analysis and geometrical intuition correctly lead you to

\[
d\sigma = \rho \, dz
\]

that's great.

If you need a little help seeing this, first consider a finite box \((L \times L \times W)\)

The charge of the slice is

\[
dq = \rho \cdot dV = \rho \cdot L^2 \, dz
\]

but it's also (because it can be treated as a flat sheet)

\[
dq = \text{"do" } A = \text{ do } L^2
\]

So we get

\[
dq = \rho \, L^2 \, dz = d\sigma \, L^2
\]

or

\[
d\sigma = \rho \, dz
\]

Which doesn't change as we take the limit \(L \to \infty\).

Note that my choice of notation "do" reminds us that the area charge density on a sheet becomes infinitesimally thick \((dz \to 0)\).
Now we can integrate.

\[ E_{\text{slab}} = \int dE_{\text{slice}} \]

\[ = \int_0^{2\pi k \rho} dz \quad \text{slice's effective area charge density} \]

\[ = \int_0^{2\pi k \rho} dz \]

\[ = 2\pi k \rho \cdot W \quad \text{We expect } E \text{ to look like } \frac{k \rho W}{\text{length}^2} \]

This is the magnitude, and the direction is just what we would expect from prev discussion.

So the final result is:

\[ \vec{E} = \vec{E}_{\text{plane}} + \vec{E}_{\text{slab}} \]

\[ = 2\pi k \rho (z^2) + 2\pi k \rho W (z^2) \]

\[ \boxed{\vec{E} = 2\pi k (\sigma + pW) \begin{cases} + \hat{z} & \rho + B \\ - \hat{z} & \rho + A \end{cases}} \]

I've included quite a lot of extra explanation along the way - for the next few problems, I'll try to just demonstrate a correct solution without too much extra stuff.
2. I'm going to use rotation that's a little more comfortable for me, and then translate back at the end.

Three spheres, centered on x-axis at \( x_0 \), \( x_1 \), and \( x_2 \).
Effective charge densities \( \rho_0, \rho_1 - \rho_0, \rho_2 - \rho_0 \) (explanation below)
Radii: \( R_0, R_1, R_2 \).

Note that we need to consider two additional spheres to make "holes" in the large sphere if we removed the \( \rho_1 \) and \( \rho_2 \) spheres, the charge density remaining there would be \( \rho_0 \), not \( 0 \).

- \( -\rho_0, R_1, x_1 \)
- \( -\rho_0, R_2, x_2 \)
This is where the \( \rho_1 - \rho_0 \) and \( \rho_2 - \rho_0 \) come from.

We'll combine all of the spheres' fields by superposition.
First consider the field due to sphere 1. (The others will have the exact same form).

Note that spheres are special—they act like point charges! (As long as you’re outside them)

So

$$\vec{E}_1 = \frac{kq_1}{r^2} \hat{r}$$

or at the point of interest,

$$\vec{E}_1(x) = \frac{kq_1}{(x-x_1)^2} \hat{x} \quad \text{for } x > x_1 + R_1$$

(Note that we’re subtracting coordinates. It does not matter whether } x \text{ or } x_1 \text{ are pos. or neg. as long as } x > x_1 + R_1.

If } x < (x_1 - R_1), \text{ direction is opposite. If } (x_1 - R_1) < x < (x_1 + R_1), \text{ we’re inside the sphere and things are different.}

What is } q_1 \text{? Since } \Psi_{1,ett} = \rho_1 - \rho_0 \text{ is uniform}

$$q_1 = (\rho_1 - \rho_0) V = (\rho_1 - \rho_0) \frac{4}{3} \pi R_1^3$$

$$\left \{ \begin{array}{c}
\text{If } p \text{ depended on coordinate } \varphi = p(\varphi) \\
\text{we would have to integrate } q = \int_{\text{object}} p(\varphi) \, dV \end{array} \right.$$
2. (cont'd)

Combine our results:

\[
\vec{E}_p = \vec{E}_0 + \vec{E}_1 + \vec{E}_2
\]

\[
= \frac{k \cdot p_0}{(x-x_0)^2} + \frac{k \cdot (p_1 - p_0)}{3 \pi R_1^3} \frac{x}{(x-x_1)^2} + \frac{k \cdot (p_2 - p_0)}{3 \pi R_2^3} \frac{x}{(x-x_2)^2}
\]

Note sign!
For \(x > 0\) and \(x < 0\),
\(x - x_0 = x + |x_0|\) appropriately.

So

\[
\vec{E}(x > 0, y = 0, z = 0) = \frac{4}{3} \pi k \left[ \frac{p R_0^3}{x^2} + \frac{(p_1 - p_0) R_1^3}{(x-x_1)^2} + \frac{(p_2 - p_0) R_2^3}{(x-x_2)^2} \right] \hat{x}
\]

or in the notation of the problem,

at point \(P\) located at \(x = d > 0\),

\[
\vec{E} = \frac{4}{3} \pi k \left[ \frac{p R^3}{d^2} + \frac{(p_1 - p) R_1^3}{(d-x_1)^2} + \frac{(p_2 - p) R_2^3}{(d-x_2)^2} \right] \hat{x}
\]

The direction of \(\vec{E}\) depends on the signs & magnitudes of the parameters.
3(a). The signs are tricky in this problem. (I was generous giving credit here.)
Regardless of the particulars of the sign, you should recognize with no calculation that \( \vec{E} \) points upwards, toward the negative charge.

I'll ignore the sign of \( \lambda \) until the end, at which point it will determine the direction of things. This means in my pictures, I'll implicitly assume \( \lambda > 0 \). I think it's the cleanest way to solve this, but it's not the only way and it's largely a matter of personal preference. Note that if you do it the "other way", you'll end up dragging around a \( \lambda \) which can be unpleasant.

Without further ado:

Draw coord system to set up integration

\[ dq = \lambda \, ds = \lambda R \, d\theta \]
3. (a) (cont'd)

Note: \( E_x = 0 \), due to symmetry

Then we're left with the \( y \) component only:

\[
E_y = \int \text{d}E_y
\]

\[
= \int \text{d}E \left( -\sin\theta \right)
\]

\[
= \int \frac{k \text{d}p}{r^2} \left( -\sin\theta \right)
\]

Substitute:

\[
k \int \frac{\lambda R \text{d}\theta}{R^2} \left( -\sin\theta \right)
\]

\[
= \frac{k \lambda}{R} \int_0^\pi \left( -\sin\theta \right) \text{d}\theta
\]

\[
= \frac{k \lambda}{R} \left[ -\cos\theta \right]_0^\pi
\]

\[
= -2\frac{k \lambda}{R}
\]

\[
E_y = -2\frac{k \lambda}{R}
\]
3. (a) (cont'd)

Let's interpret this,

\[ \vec{E} = \frac{-2k \lambda}{R} \hat{y} \]

and

\[ |\vec{E}| = \frac{2k |\lambda|}{R} \]

If \( \lambda < 0 \), as in this problem, the direction of \( \vec{E} \) is \( +\hat{y} \), and the magnitude is

\[ |\vec{E}| = |E| = \frac{2k |\lambda|}{R} \]

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**Note:** Can't drop the abs value!
3.(b) Break into a series of semi-circles w/ fields

\[ dE_y = \frac{-2k \, d\lambda}{R} \]

The idea is very much the same as in #1.
Find charge of semi-circ two ways (bend it into a rectangle)

\[ d\lambda = \sigma \, dA = \sigma \, (\pi R \, dR) \]
\[ d\lambda = d\lambda (\pi R) \]

so

\[ d\lambda = \sigma \, dR \]

Integrate (only have y component)

\[ E_y = \int dE_y (\text{semi-circle}) = \int \frac{-2k \, d\lambda}{R} = \int \frac{-2k \, (\sigma \, dR)}{R} \]

\[ E_y = -2k \sigma \int_{R_1}^{R_2} \frac{dR}{R} \]

\[ E_y = -2k \sigma \ln \left( \frac{R_2}{R_1} \right) \]

\[ \text{this is sufficient, or could state} \]
\[ E = 2k \sigma \ln \left( \frac{R_2}{R_1} \right) \]
\[ \text{and } E \text{ points down & } \sigma < 0 \]
3(b). (con'd)

Now take the limit $R_1 \to 0$.

$$\lim_{R \to 0} \mathbf{E}_y = \lim_{R \to 0} \left( -2k\sigma \right) \ln \left( \frac{R_0}{R_1} \right)$$

$$= -2k\sigma \ln \infty$$

$$= \pm \infty$$

↑ depends on sign of $\sigma$

For $\sigma < 0$,

$$\left| \mathbf{E}_y \to +\infty \right|$$

Does this make sense?

Yes, we're sitting inside a sheet of charge in that limit. It's the same idea as sitting on a point charge —

$$|\mathbf{E}| \to \infty.$$