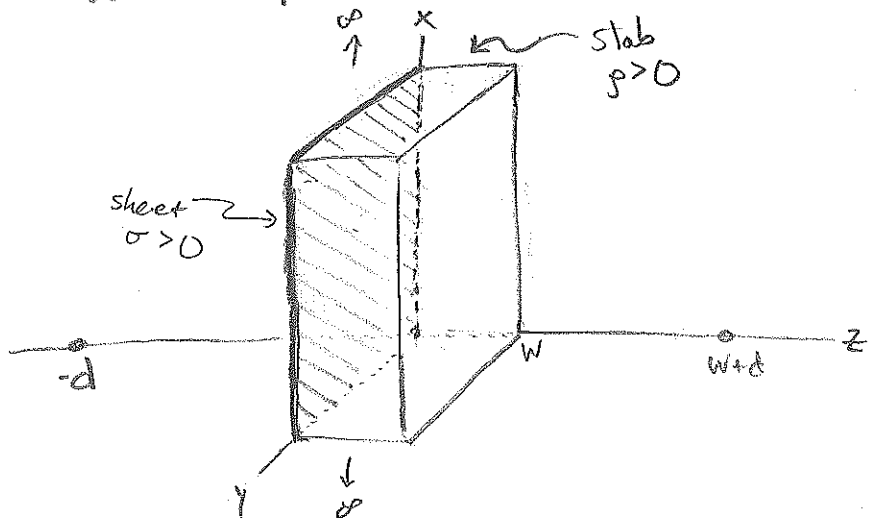
Phy 162 - Quiz 2
1/28/2015

Solutions

1. First, set up a coord. system:
(and forcing yourself to draw the physical system adds clarity for yourself & readers of your work)



Note that the slab & plane are ∞ in x+y directions (as I've chosen them)

Think qualitatively first (not required for credit, but a useful exercise) What kind of fields do we expect outside of the slab-plane system? Symmetry tells us that the should only have $\hat{z}$ components (imagine reflecting the system through the x-z or y-z planes — the physical system doesn't change, so the field should not). Symmetry also tells us that $\vec{E}(x,y,z) = \vec{E}(z)$, that is, $\vec{E}$ should not have a functional dependence on x & y (imagine translating the system in the x-y plane — again, same physical system ⇒ same fields). So we conclude that $\vec{E} = E(z)\,\hat{z}$, and we could now usefully apply Gauss's law, as you do in the next chapter.

It is also evident that for $q > 0$ and $\sigma > 0$ the fields should point away from the system.

But that is unnecessary here, since we're given the $\vec{E}$ field due to an ∞ plane, which is information enough. Note that in the end, because of the ∞ plane distribution, $\vec{E}$ does not even depend on z — it's uniform on a given side of the plane.

1. (cont'd)

Now, back to calculating.

Use superposition:

$$\vec{E} = \vec{E}_{plane} + \vec{E}_{slab}$$

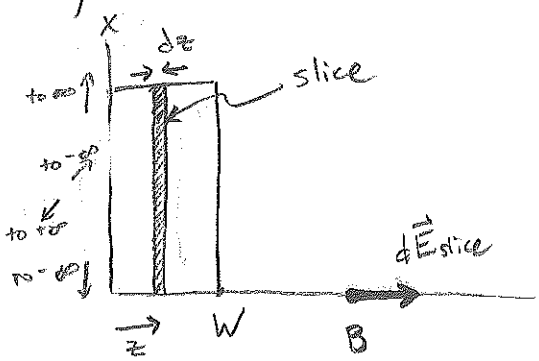
We're given

$$\left\|\vec{E}_{plane}\right\| = 2\pi k\sigma$$

(more commonly written $\sigma/2\varepsilon_0$, as is readily obtained via Gauss's law)

and we know it points away and $\perp$ to plane.

We'll use this idea to calculate $\vec{E}_{slab}$ by slicing the slab into infinitesimally thick planes:



$$\vec{E}_{slab} = \int_{\substack{\text{all}\\\text{slices}}} d\vec{E}_{slice} \xrightarrow{\text{all } d\vec{E} \text{ point in same dir}} \left|\vec{E}_{slab}\right| = \int \left|d\vec{E}_{slice}\right|$$

Which we can write as

$$E_{slab} = \int dE_{slice}$$

The fields due to the slices look like

$$dE_{slice} = 2\pi k\, d\sigma$$

This might often be written as just "σ", but I think this will be clearer here.

1. (cont'd)

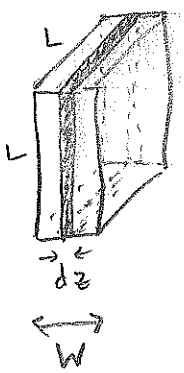
What is $d\sigma$?

If a combination of dimensional analysis & geometric intuition correctly lead you to

$$d\sigma = \rho\, dz$$

that's great.

If you need a little help seeing this, first consider a <u>finite</u> box $(L \times L \times W)$



The charge of the slice is

$$dq = \rho \cdot dV$$
$$= \rho \cdot L^2 dz$$

but it's also (because it can be treated as a flat sheet)

$$dq = \text{"}d\sigma\text{"} A$$
$$= d\sigma\, L^2$$

So we get

$$dq = \rho L^2 dz = d\sigma L^2$$

or

$$d\sigma = \rho\, dz$$

Which doesn't change as we take the limit $L \to \infty$.

Note that my choice of notation "$d\sigma$" reminds us that the area charge density on a sheet becomes infinitesimal as the sheet becomes infinitesimally thick ($dz \to 0$).

1. (cont'd)

Now we can integrate.

$$E_{slab} = \int_{\substack{\text{all}\\\text{slices}}} dE_{slice}$$

$$= \int_{\substack{\text{all}\\\text{slices}}} 2\pi k \, d\sigma$$

slice's effective area charge density

$$= \int_0^W 2\pi k \rho \, dz$$

$$= 2\pi k \rho \cdot W$$

Note this is dimensionally correct! We expect E to look like k times $\left(\frac{\text{charge}}{\text{length}^2}\right)$

This is the magnitude, and the direction is just what we would expect from prev discussion.

So the final result is.

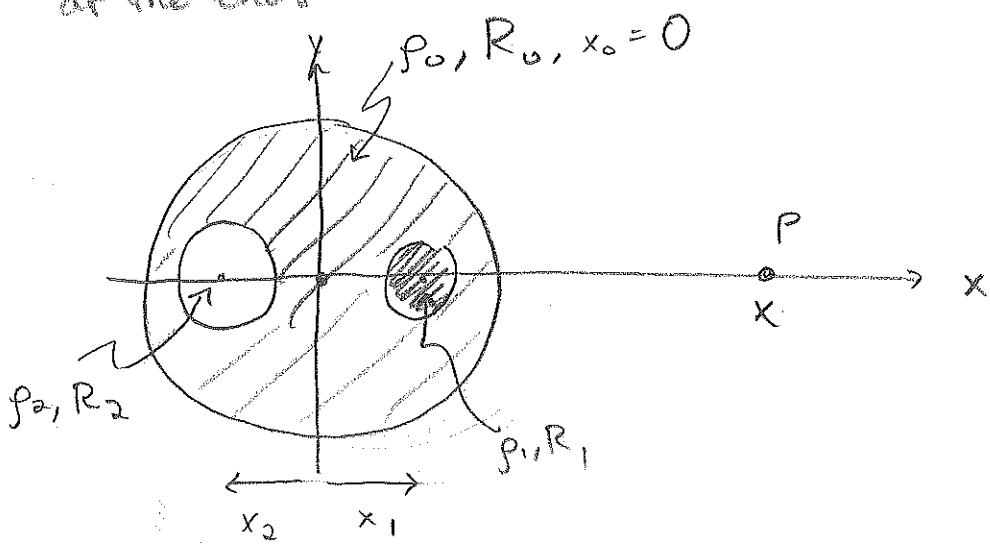
$$\vec{E} = \vec{E}_{plane} + \vec{E}_{slab}$$

$$= 2\pi k\sigma(\pm\hat{z}) + 2\pi k\rho W(\pm\hat{z})$$

$$\boxed{\vec{E} = 2\pi k(\sigma + \rho W) \begin{cases} +\hat{z} & @ \text{ pt } B \\ -\hat{z} & @ \text{ pt } A \end{cases}}$$

I've included quite a lot of extra explanation along the way — for the next few problems, I'll try to just demonstrate a correct solution without too much extra stuff.

2. I'm going to use notation that's a little more comfortable for me, and then translate back at the end.



Three spheres, centered on x axis at x_0, x_1, and x_2.
Effective charge densities ρ_0, $\rho_1 - \rho_0$, $\rho_2 - \rho_0$ (explanation below)
Radii R_0, R_1, R_2.

Motivation for "effective charge densities"

Note that we need to consider two additional spheres to make "holes" in the large sphere (if we removed the ρ_1 and ρ_2 spheres, the charge density remaining there would be 0, not ρ_0.

- $-\rho_0$, R_1, x_1
- $-\rho_0$, R_2, x_2

This is where the $\rho_1 - \rho_0$ and $\rho_2 - \rho_0$ come from.

We'll combine all of the spheres' fields by superposition.

2. (cont'd)

First, consider the field due to sphere 1.
(the others will have the exact same form).

Note that spheres are special — they act like point charges! (As long as you're outside them)

So

$$\vec{E}_1 = \frac{kq_1}{r^2}\hat{r}$$

or at the point of interest,

$$\vec{E}_1(x) = \frac{kq_1}{(x-x_1)^2}\hat{x} \quad \text{for } x > x_1 + R_1$$

(Note that we're subtracting coordinates. It does not matter whether x or x_1 are pos. or neg. as long as $x > x_1 + R_1$. If $x < (x_1 - R_1)$, direction is opposite. If $(x_1 - R_1) < x < (x_1 + R_1)$, we're inside the sphere & things are different.)

What is q_1?
Since $\rho_{1,eff} = \rho_1 - \rho_0$ is uniform

$$q_1 = (\rho_1 - \rho_0)V = (\rho_1 - \rho_0)\frac{4}{3}\pi R_1^3$$

(If ρ depended on coordinate $\rho = \rho(\vec{r})$ we would have to integrate $q = \int_{\text{object}} \rho(\vec{r})\, dV$)

2. (cont'd)

Combine our results:

$$\vec{E}\Big|_P = \vec{E}_0 + \vec{E}_1 + \vec{E}_2$$

$$= \frac{k \cdot \rho_0 \frac{4}{3}\pi R_0^3}{(x-x_0)^2}\hat{x} + \frac{k \cdot (\rho_1 - \rho_0)\frac{4}{3}\pi R_1^3}{(x-x_1)^2} + \frac{k(\rho_2 - \rho_0)\frac{4}{3}\pi R_2^3}{(x-x_2)^2}$$

Note sign!
For $x>0$ and $x_2<0$
$x - x_2 = x + |x_2|$
appropriately

So

$$\vec{E}(x>0, y=0, z=0) = \frac{4}{3}\pi k\left[\frac{\rho_0 R_0^3}{x^2} + \frac{(\rho_1-\rho_0)R_1^3}{(x-x_1)^2} + \frac{(\rho_2-\rho_0)R_2^3}{(x-x_2)^2}\right]\hat{x}$$

or in the notation of the problem,
at point P located at $x = d > 0$,

$$\boxed{\vec{E} = \frac{4}{3}\pi k\left[\frac{\rho R^3}{d^2} + \frac{(\rho_1-\rho)R_1^3}{(d-x_1)^2} + \frac{(\rho_2-\rho)R_2^3}{(d-x_2)^2}\right]\hat{x}}$$

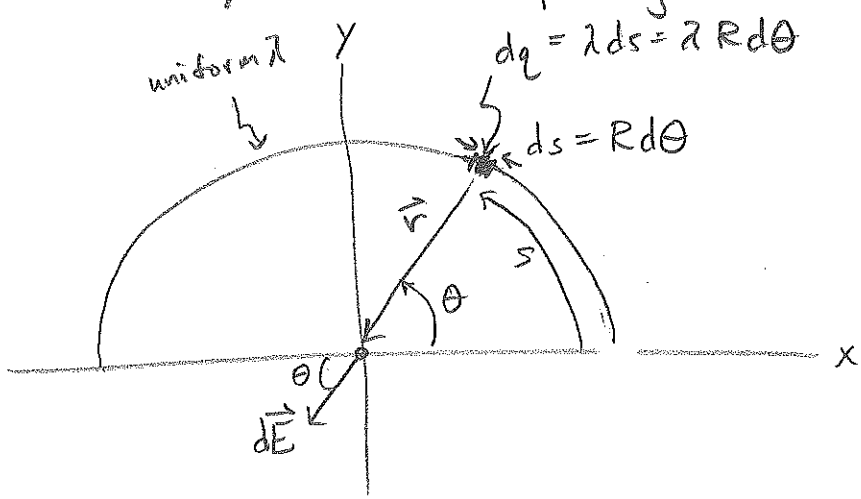
The <u>direction</u> of $\vec{E}$ depends on the signs & magnitudes of the parameters.

3(a). The signs are tricky in this problem.
(I was generous giving credit here)
Regardless of the particulars of the sign,
you should recognize with no calculation
that $\vec{E}$ points <u>upwards</u>, toward the negative
charge.

I'll ignore the sign of λ until the end,
at which point it will determine the direction of things.
This means in my pictures, I'll implicitly assume $\lambda > 0$.
I think it's the cleanest way to solve this,
but it's not the only way and it's largely a
matter of personal preference. Note that if
you do it the "other way", you'll end up dragging
around a $|\lambda|$ which can be unpleasant.

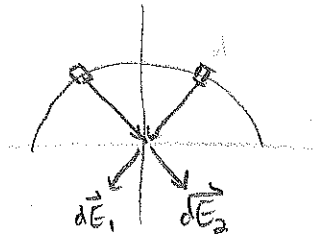
Without further ado:

Draw coord system to set up integration



3.(a). (cont'd)

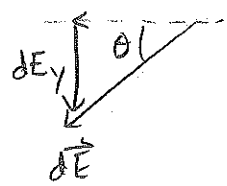
Note $E_x = 0$, due to symmetry



Then we're left with the y component only:

$$E_y = \int_{\text{semicirc}} dE_y$$

$$= \int_{sc} dE\,(-\sin\theta)$$



$$= \int_{sc} \frac{k\,dq}{r^2}(-\sin\theta)$$

Substitute.

$$= k\int_0^{\pi} \frac{(\lambda R\,d\theta)}{R^2}(-\sin\theta)$$

$$= \frac{\lambda k}{R}\int_0^{\pi}(-\sin\theta)\,d\theta$$

not necessary, but often helpful

all constants with respect to integration variable

$$= \frac{2\lambda k}{R}\int_0^{\pi/2}(-\sin\theta)\,d\theta$$

$$= \frac{2\lambda k}{R}\cos\theta\Big|_0^{\pi/2}$$

$$E_y = \frac{-2\lambda k}{R}$$

3. (a) (cont'd)

Let's interpret this.

$$\vec{E} = \frac{-2k\lambda}{R}\hat{y}$$

and

$$|\vec{E}| = \frac{2k|\lambda|}{R}$$

If $\lambda < 0$, as in this problem,

the direction of $\vec{E}$ is $\underline{+\hat{y}}$, and

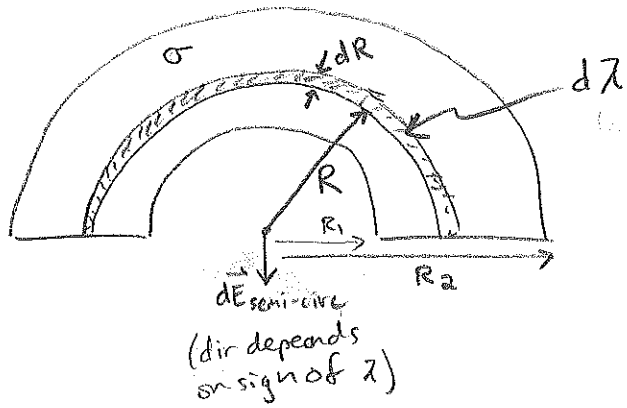
the magnitude is

$$\boxed{E = |\vec{E}| = \frac{2k|\lambda|}{R}}$$

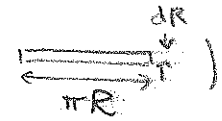
can't drop the abs value!

3.(b) Break into a series of semi circles w/ fields

$$dE_y = \frac{-2k\,d\lambda}{R}$$



The idea is very much the same as in #1.

Find charge of semi-circ two ways (bend it into a rectangle 

)

- $dq = \sigma\, dA = \sigma\,(\pi R\, dR)$
- $dq = d\lambda\,(\pi R)$

so

$$d\lambda = \sigma\, dR$$

Integrate (only have y component)

$$E_y = \int_{\substack{\text{semi-}\\\text{annulus}}} dE_{y\,(\text{semi-circle})} = \int \frac{-2k\,d\lambda}{R} = \int \frac{-2k\,(\sigma\, dR)}{R}$$

$$= -2k\sigma \int_{R_1}^{R_2} \frac{dR}{R}$$

$$\boxed{E_y = -2k\sigma \ln\left(R_2/R_1\right)}$$

← this is sufficient, or could state

$$E = 2k|\sigma| \ln\left(R_2/R_1\right)$$

and $\vec{E}$ points down b/c $\sigma < 0$

3(b). (cont'd)

Now take the limit $R_1 \to 0$.

$$\lim_{R_1 \to 0} E_y = \lim_{R_1 \to 0} (-2k\sigma) \ln(R_2/R_1)$$

$$= -2k\sigma \ln \infty$$

$$= \mp \infty$$

depends on sign of σ

For $\sigma < 0$,

$$\boxed{E_y \longrightarrow +\infty}$$

Does this make sense?

Yes, we're sitting inside a sheet of charge in that limit. It's the same idea as sitting on a point charge — $|\vec{E}| \to \infty$.