From Chapter 26, you know the \( \vec{E} \) field of some basic symmetric objects:

- point
- finite or infinite plane
- finite or infinite line

Can get \( \vec{E} \) field of other objects by integration and superposition, examples in both and in homework.

Can also solve some problems that have gaps or holes by superimposing objects with negative charge densities with objects with positive charge densities.

Some examples:

1. What is \( \vec{E} \) at \( P \) on symmetry line of gap \( g \) with \( \rho \). Same physically.

2. Sphere of volume density \( \rho \) has hole carved out of radius \( R/2 \) as shown. What is \( \vec{E} \) at \( P \). Same as two spheres, one full of radius \( R/2 \), one of radius \( R/4 \), charge density \( -\rho \).
Join to Chapter 27: **Gauss's Law**

(1) Method to compute \( \vec{E} \) quickly inside and outside symmetrical objects with inhomogeneous (non-constant) charge densities

- Earth with water, rocky, iron layers
- Life or ion
- Cylindrical with hole, layered

(2) Same as Coulomb's law for static arrangements of charges but is more general since valid for moving charges no matter how fast, compatible with special relativity.

Gauss's Law is the first of four Maxwell equations that give unified quantitative description of all classical (non-quantum) electromagnetic phenomena.

- Electric charge produces \( \vec{E} \) field
  \[ \oint \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0} \]

- Currents produce \( \vec{B} \) field

- Changing \( \vec{E} \) produces \( \vec{B} \)

- Changing \( \vec{B} \) produces \( \vec{E} \)

\[ \oint \vec{B} \cdot d\vec{l} = \mu_0 I + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \]

\[ \oint \vec{E} \cdot d\vec{l} = -\frac{1}{c^2} \frac{\partial \Psi}{\partial t} \]

(3) Useful for getting insight about charge inside and on conductors

- Metall sphere with hole or cavity

- What is \( \vec{E} \) everywhere?
Gauss's law is one of the more difficult and abstract results in intro physics and is notoriously confusing for students seeing it for the first time. It roughly says that some kind of average of the electric field \( \vec{E}(x,y,z) \) over the surface of some imaginary closed bubble tells you something about the total charge enclosed in the bubble.

\[ \int_{\text{surface}} \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0} \]

To see qualitatively why there may be some connection, think of black box with unknown contents and you measure the \( \vec{E} \) field at many points outside the box.

\[ \text{what's inside?} \quad \text{what's inside?} \quad \text{what's inside the box?} \]
By considering these three boxes, you can see that if most \( \vec{E} \) vectors point outward, then surface of box should be net positive charge inside, if most point inwards then net negative charge inside, if equal amounts pointing in and out, no net change.

These ideas lead to specific mathematical relation between \( \vec{E} \) own surface of bubble and total charge enclosed by the bubble. Key quantity is called \( \text{Flux of electric field through some area} \).

Consider some small region of space such that \( \vec{E} \) approximately constant in magnitude and direction, can always arrange this since \( \vec{E} \) varies continuously. Consider some small planar area, shape doesn't matter but let's consider a square.

\[
\vec{A} = L \hat{n}
\]

\( \hat{n} \) = normal unit vector

\[
\int \int \vec{E} \cdot \vec{A} \text{ dA} \]

Can assign vector to square \( \vec{A} \) whose magnitude is area \( A \) of square and whose direction is perpendicular to square then can obtain a number called "the Flux \( \Phi \) of \( \vec{E} \) through \( \vec{A} \)" by simply calculating the dot product

\[
\Phi = \vec{E} \cdot \vec{A} \]

(\( \Phi \)) capital Greek phi

(lower case is \( \phi \))
Then $\Phi$ gives a sense of how much $\vec{E}$ points into square

\[ \Phi = (\vec{E} \hat{i}) \cdot (\vec{A} \hat{j}) = AE \]

Square tilted by angle $\theta$ from vertical

$\vec{n} = \langle \cos \theta, \sin \theta \rangle$

\[ \Phi = \vec{E} \cdot \vec{n} \]

$= (\vec{E} \hat{i}) \cdot (\vec{A} \hat{j} \cos \theta + \hat{k} \sin \theta)$

$= EA \cos \theta$

For $\theta = \pi/2$, square is horizontal and no $\vec{E}$ passes through $\vec{A}$

$\Phi = 0$

For isolated square, there are two $\vec{A}$'s pointing in opposite directions, which one to choose? If one is part of surface, then choose $\vec{A}$ that points from interior to exterior of bubble

For cube, all $\vec{A}$'s point away from interior
Discuss some cliché questions related to flux $\mathbf{E}$ through some area $\mathbf{A}$.
For given electric field filling all of space, given closed bubble that defines some surface $S$, we can imagine carving surface into tines and tines areas, say squares. When squares are tiny enough, $\vec{E}$ is approximately constant over the square, and then we can calculate the flux $\Phi_i = \vec{E}_i \cdot \vec{A}_i$ for $\vec{E}$ over the $i$th area $\vec{A}_i$.

With our convention that $\vec{A}_i$ points away from interior, $\Phi_i > 0$ if $\vec{E}$ points away from interior; $\Phi_i < 0$ if $\vec{E}$ points into interior.

Next, we can add up all the fluxes over all the small areas that make up the surface to get the total flux of $\vec{E}$ into the bubble

$$\Phi = \sum_i \Phi_i = \sum_i \vec{E}_i \cdot \vec{A}_i$$

We can then imagine taking the usual calculus limit, in which the areas become infinitesimally small $d\vec{A}_i$; and infinitely many and the sum turns into an integral

$$\Phi = \lim_{n \to \infty} \sum \vec{E}_i \cdot d\vec{A}_i \to \int_S \vec{E} \cdot d\vec{A} = \Phi$$

$C$ indicates surface or bubble of interest.
This is called a surface integral over the surface $S$ of the bubble because we are using information known only at points on the surface, $\vec{E}$ and $d\vec{A}$.

This is a scary or confusing integral and is generally difficult to evaluate for arbitrary bubbles and general $\vec{E}$ fields. But for bubbles with high symmetry:

- Sphere
- Cylinder
- Plane

and for $\vec{E}$ fields that have the same symmetry, the integrals are trivial and can be done in your head, they all have the form: "magnitude of $\vec{E}$" x surface area of bubble.

Then Gauss's law can be written as follows:

Given an arbitrary closed surface $S$, given some electric field filling all of space $\vec{E}(x, y, z)$, then

$$\Phi = \int \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

$Q_{\text{enc}}$ = total charge enclosed by bubble
$\varepsilon_0$ = vacuum permittivity defined by

$$K = \frac{1}{4\pi \varepsilon_0}$$
Extremely important for you to realize that $\vec{E}$ field in Gauss's law is $\vec{E}$ field due to all changes in the universe, inside and outside the bubble. For example:

\begin{align*}
\int_{S} \vec{E} \cdot d\vec{A} &= \frac{(-5e + 4e + 6e)}{\varepsilon_0} \\
\text{even though } \vec{E} \text{ arises from all indicated sources.}
\end{align*}
Some classic questions on Gauss's law

\[ \Phi_1 = \Phi \]
\[ \Phi_2 = B \]

What is \( q_1 \) and \( q_2 \).
Let's take our time and work through several examples to see what Gauss's theorem means, how to get insight.

1. Consider point charge with charge \( Q > 0 \). \( \vec{E} \) is radial, points everywhere away from charge.

Consider spherical bubble of radius \( R \).

\[ \begin{align*}
\vec{E} & \text{ same everywhere on surface} \\
\vec{E} \parallel \text{ every where on surface}
\end{align*} \]

Every little area has normal also pointing radially from charge interior to exterior, so also radially.

Then, \[ \vec{E} \cdot dA = EdA \]

since \( \vec{E} \parallel dA \)

since \( E = E(\hat{r}) \)

\[ I = \oint \vec{E} \cdot dA = \oint EdA = E \cdot \int dA = E \cdot 4\pi R^2 \]

so flux is unknown \( E \) magnitude (that varies with \( R \)) times surface area of sphere.

Gauss's law then says:

\[ I = \oint \vec{E} \cdot dA = E \cdot 4\pi R^2 = \frac{Q_{\text{enc}}}{\varepsilon_0} = \frac{Q}{\varepsilon_0} \]

\[ \Rightarrow E = \frac{Q}{4\pi \varepsilon_0} \frac{1}{R^2} = \frac{KQ}{R^2} \quad \text{Coulomb's law} \]
Note the argument breaks down if charge not at center of sphere so $\vec{E}$ and sphere don't have some symmetry.

Now $\vec{E}$ and $dA$ not everywhere parallel. Flux is still $\frac{Q}{\varepsilon_0}$ by Gauss's theorem but now we can't solve for $\vec{E}$ since $\vec{E}$ no longer constant over surface of sphere.

**Example 2**  
Cylinder in uniform electric field produced by planar capacitor.

$\vec{E}$ everywhere the same. No charge inside this cylinder bubble so we expect $\Phi = \int_{\text{cylinder}} \vec{E} \cdot d\vec{A} = 0$.

Let's verify directly.

\[ \Phi_{\text{cylinder}} = \Phi_A + \Phi_B + \Phi_C \]
\[ \Phi_A = (E \hat{i}), \quad A(\hat{\mathbf{i}}) = -AE \quad \text{point in opposite direction} \]

\[ \Phi_B = \int \mathbf{E} \cdot d\mathbf{A} \quad \text{but} \quad \mathbf{E} \perp d\mathbf{A} \quad \text{everywhere on surface} \]

of curved wall of cylinder \quad \Rightarrow \Phi_B = 0 \quad \text{so} \quad \mathbf{E} \cdot d\mathbf{A} = 0 \]

\[ \Phi_C = (E \hat{i}) \cdot (A \hat{i}) = EA \]

\[ \Phi_{\text{Total}} = \Phi_A + \Phi_B + \Phi_C = -AE + 0 + AE = 0 = \frac{Q_{\text{enc}}}{\varepsilon_0} \]

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Example 3: Flux through cube with charge at center

\[ \Phi = \sum E \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0} \]

But here, unlike sphere, we can't evaluate \[ E \cdot d\mathbf{A} \] easily since \( E \) changes magnitude and direction over face of each cube, can not write this as constant magnitude \( E \) x surface area because cube lacks symmetry of \( E \) field, which has spherical symmetry.
If charge is at center of cube, we can calculate flux through each of the six faces because, by symmetry, flux is same for each face.

\[
\frac{Q}{\varepsilon_0} = \Phi_{\text{total}} = \oint_{\text{cube}} \textbf{E} \cdot d\textbf{A} = 6 \cdot \oint_{\text{face}} \textbf{E} \cdot d\textbf{A} = 6 \Phi_{\text{face}}
\]

\[
\Phi_{\text{face}} = \frac{Q}{6\varepsilon_0}
\]

For practice working with vectors and flux, let's confirm this by direct integration.

Let's put charge \( Q \) at origin. Then \( Q \) is distance \( \frac{1}{2} \) from each face in cube of side \( L \) so let's compute flux through face in plane \( z = \frac{1}{2} \) whose vertices are

\[
\left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)
\]
We can divide the face into little infinitesimal squares centered on \((xy)\) with sides \(dx\), \(dy\). Area of square is \(dx\cdot dy\) and its vector must be \(\hat{F} = (0,0,1)\) since normal vector points from inside to outside of the cube, so

\[
\vec{dA} = dx \cdot dy \cdot (0,0,1) \quad \text{for all} \quad -\frac{L}{2} < x < \frac{L}{2}, -\frac{L}{2} < y < \frac{L}{2}
\]

At coordinate \((x,y,\frac{L}{2})\), what is \(\vec{E}\) field? By Coulomb's law,

\[
\vec{E} = \frac{KQ}{(x^2+y^2+(\frac{L}{2})^2)^{3/2}} \cdot \hat{r} = \frac{(x,y,\frac{L}{2})}{\sqrt{x^2+y^2+(\frac{L}{2})^2}}
\]

\[
= KQ \cdot \frac{(x,y,\frac{L}{2})}{(x^2+y^2+(\frac{L}{2})^2)^{3/2}}
\]

The flux \(\vec{E} \cdot \vec{dA}\) through little square at \((x,y)\) is

\[
\vec{dJ}(xy) = \vec{E}(xy,\frac{L}{2}) \cdot \vec{dA}
\]

\[
= KQ \frac{(x,y,\frac{L}{2})}{(x^2+y^2+(\frac{L}{2})^2)^{3/2}} \cdot dx \; dy \; (0,0,1)
\]

\[
= \frac{(1/2)KQ}{(x^2+y^2+(\frac{L}{2})^2)^{3/2}} \; dx \; dy
\]

Total flux through square is then:
\[ \Phi_0 = \int_{-\frac{L}{2}}^{\frac{L}{2}} \left[ \int_{-\frac{L}{2}}^{\frac{L}{2}} dy \right] dF(x,y) \]

\[ = \frac{K\ell}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \int_{-\frac{L}{2}}^{\frac{L}{2}} dy \left( x^2 + y^2 + \left( \frac{L}{2} \right)^2 \right)^{-3/2} \]

You can verify that the integral \( I \) has physical units of

\[ \frac{dx}{L} \cdot \frac{dy}{L} = \frac{1}{L} \]

and, generally, it is useful to change variables so that the integral can be written as a purely numerical integral (no physical units) times symbols that carry the physical units. The advantage of having a non-symbolic integral is that, if necessary, one can try to approximate it numerically without having to choose particular values for symbols.

Here, it is straightforward to eliminate the various \( L/2 \) symbols by changing variables to two dimensionless lengths \( u \) and \( v \):

\[ x = \frac{1}{2} u, \quad dx = \frac{1}{2} du \quad x = -\frac{L}{2} \rightarrow u = -1 \]

\[ x = \frac{L}{2} \rightarrow u = +1 \]

\[ y = \frac{L}{2} v, \quad dy = \frac{1}{2} dv \quad y = -\frac{L}{2} \rightarrow v = -1 \]

\[ y = \frac{L}{2} \rightarrow v = +1 \]
The integral \( I \) then becomes:

\[
I = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} (x^2 + y^2 + \left(\frac{1}{2}\right)^2)^{-\frac{3}{2}} \, dx \, dy
= \frac{\pi}{2} \int_{-1}^{1} \int_{-1}^{1} (u^2 + v^2 + 1)^{-\frac{3}{2}} \, du \, dv
\]

where \( u = x - \frac{1}{2}, \ \text{and} \ \ v = y - \frac{1}{2} \).

We can then approximate \( J \) numerically, e.g., in Mathematica type:

\[
\text{NIntegrate}[
(u^2 + v^2 + 1)^{-\frac{3}{2}},
\{u,-1,1\}, \{v,-1,1\}
\]

which gives the approximate value \( 2.094 \). But Mathematica can evaluate \( J \) analytically like this:

\[
\text{Integrate}[
(u^2 + v^2 + 1)^{-\frac{3}{2}},
\{u,-1,1\}, \{v,-1,1\}
\]

which gives the exact symbolic answer

\[
J = \frac{2\pi}{3} \approx 2.094
\]
We conclude that:

\[ \Phi = \text{flux through side of cube of length } L \text{ with } \]
\[ \text{charge } Q \text{ at center} \]
\[ = \frac{KQL \times I}{2} = \frac{KQL}{2} \left( \frac{2}{L} J \right) \]
\[ = \frac{KQL}{2} \left( \frac{2}{L^3} \right) \]
\[ = \frac{Q}{6\epsilon_0}, \text{ in agreement with Gauss's law} \]

Can you adapt the symmetry argument to deduce the flux through a side of a cube when the charge is placed at a vertex, instead of at the cube's center?

\[ \Phi \]

What is flux \( \Phi \) through front face?