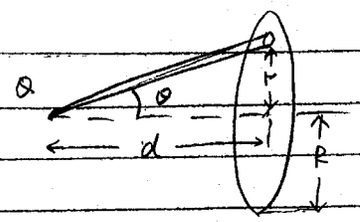
PHY 162
Assignment 3
Solutions

1.



To calculate the flux Φ through the circle due to Q, we divide the circle into concentric infinitesimal rings. Consider a ring with radius r and width dr.

$$d\Phi = \vec{E}\cdot\vec{n}\,dA = E\,dA\cos\theta$$

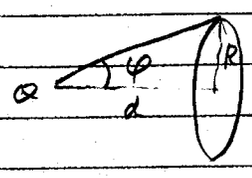
in which $E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2+d^2}$, $dA = 2\pi r\,dr$, $\cos\theta = \frac{d}{\sqrt{r^2+d^2}}$.

$$\Rightarrow \quad d\Phi = \frac{Qd}{4\pi\varepsilon_0}\frac{2\pi r\,dr}{(r^2+d^2)^{3/2}}$$

$$\Phi = \int_{r=0}^{R}\frac{Qd}{4\pi\varepsilon_0}\frac{2\pi r\,dr}{(r^2+d^2)^{3/2}} = \frac{Qd}{2\varepsilon_0}\left(\frac{1}{d}-\frac{1}{\sqrt{d^2+R^2}}\right)$$

$$= \frac{Q}{2\varepsilon_0}\left(1-\frac{d}{\sqrt{d^2+R^2}}\right)$$

Note that $\frac{d}{\sqrt{d^2+R^2}}$ is the cosine of the angle made by the x-axis and the line connecting Q and the edge of the circle, φ. Therefore:



$$\Phi = \frac{Q}{2\varepsilon_0}(1-\cos\varphi) = \frac{Q}{2\varepsilon_0}\left(1-\frac{d}{\sqrt{d^2+R^2}}\right)$$

One limit to test: does flux $\Phi \to 0$ as $R \to 0$? This should be the case since the flux through zero area is zero. But

$$\lim_{R\to 0}\Phi = \lim_{R\to 0}\frac{Q}{2\varepsilon_0}\left(1-\frac{d}{\sqrt{d^2+R^2}}\right) = \frac{Q}{2\varepsilon_0}\left(1-\frac{d}{d}\right) = 0 \checkmark$$

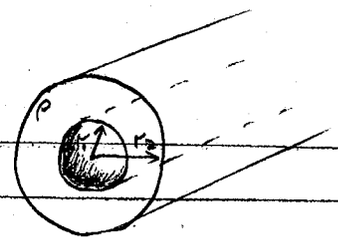
Another limit to test: $d \to \infty$, then $\Phi = 0$ since ~~fixed area~~ $\vec{E} \to 0$ as $d \to \infty$

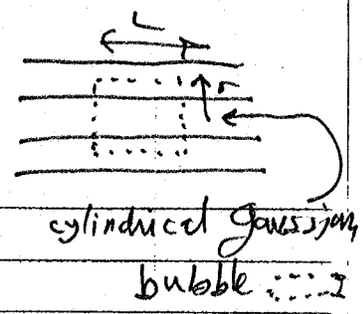
3rd limit to test: $R \to \infty$, then $\Phi \to \frac{1}{2}$ flux through sphere $= \frac{1}{2}\left(\frac{Q}{\varepsilon_0}\right) = \frac{Q}{2\varepsilon_0}$ ✓

4th limit to test: does $\Phi \to 0$ as $Q \to 0$

answer is yes in all cases

2.





Due to the symmetrical feature of the structure, we can apply Gauss's Law to calculate the E-field. We construct Gaussian surfaces in the shape of a cylinder of radius r and length l and has the same axis as the charge cylinder, then:

$$\oint_S E_n \, dA = \frac{1}{\varepsilon_0} Q_{enclosed}$$

$$2\pi r l E = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$\Rightarrow E = \frac{Q_{enclosed}}{2\pi r l \varepsilon_0}$$

Note that we neglect the end areas because no flux crosses them due to symmetry.

(1) $r < r_1$: $\quad E_1 = \frac{0}{2\pi r l \varepsilon_0} = 0$

(2) $r_1 < r < r_2$:

$$Q_{enclosed} = \rho V = \rho \pi l (r^2 - r_1^2)$$

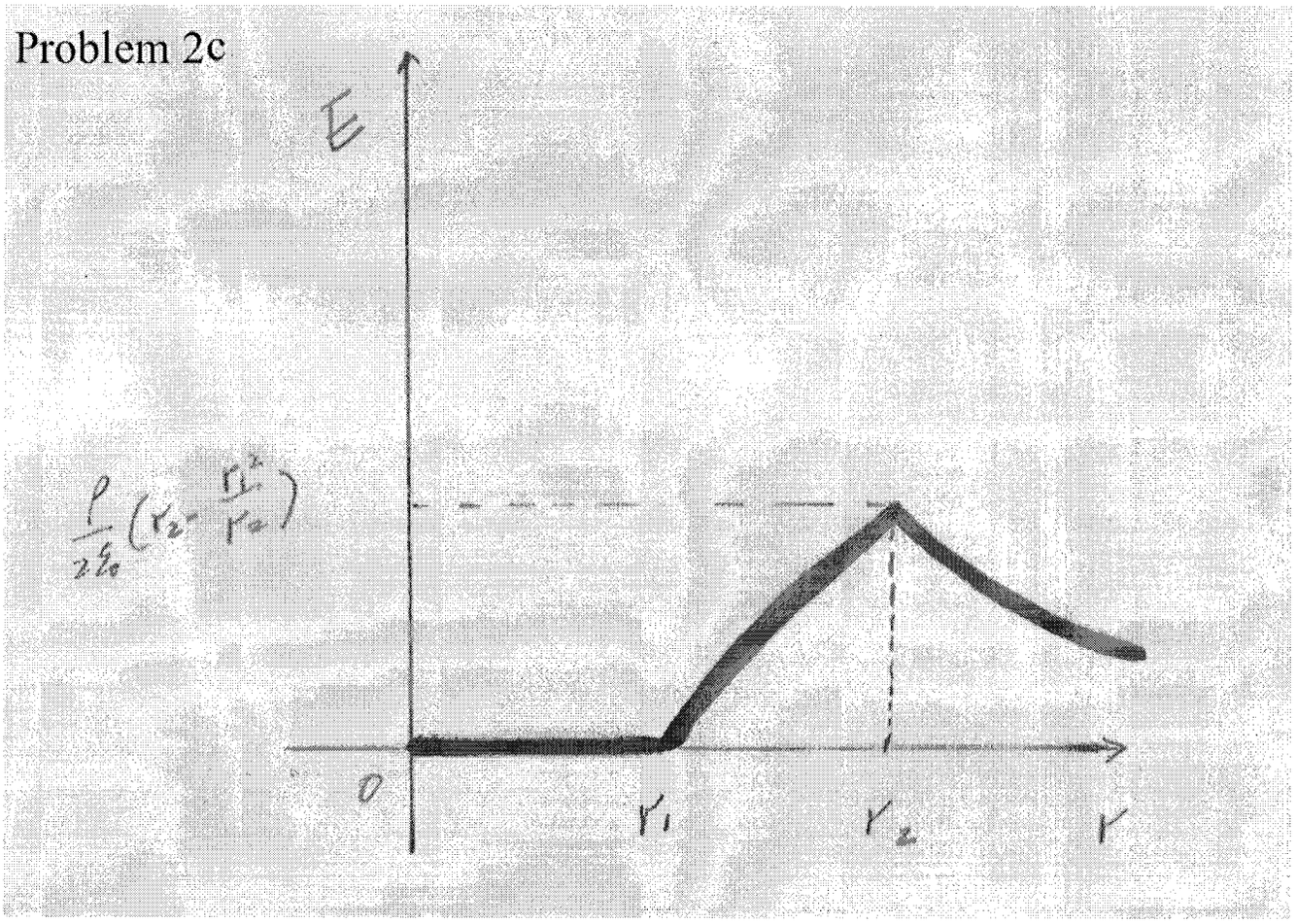
$$\Rightarrow E_2 = \frac{\rho \pi l (r^2 - r_1^2)}{2\pi r l \varepsilon_0} = \frac{\rho (r^2 - r_1^2)}{2\varepsilon_0 r}$$

(3) $r > r_2$:

$$Q_{enclosed} = \rho V = \rho \pi l (r_2^2 - r_1^2)$$

$$\Rightarrow E_3 = \frac{\rho \pi l (r_2^2 - r_1^2)}{2\pi r l \varepsilon_0} = \frac{\rho (r_2^2 - r_1^2)}{2\varepsilon_0 r}$$

Problem 2c



3. (a) Due to symmetry, we know that the E-field at any point inside the sphere is directing along the radius away from the center, and we can apply Gauss's Law to calculate the E-field.

For a spherical Gaussian surface of radius r centered at the origin:

$$\oint_S E_n \, dA = 4\pi r^2 E = \frac{Q_{enclosed}}{\varepsilon_0}$$

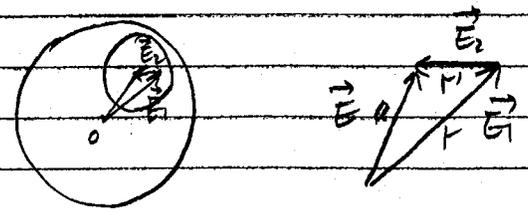
and:

$$Q_{enclosed} = \frac{4}{3}\rho\pi r^3$$

$$\Rightarrow E = \frac{\frac{4}{3}\rho\pi r^3}{4\pi r^2 \varepsilon_0} = \frac{1}{3\varepsilon_0}\rho r$$

$$\Rightarrow \vec{E} = \frac{\rho r}{3\varepsilon_0}\hat{r}$$

(b).



We can consider the sphere with cavity as two spheres of equal positive and negative charge densities.

E-field due to the "positively charged sphere":

$$\vec{E}_1 = \frac{\rho r}{3\varepsilon_0}\hat{r}$$

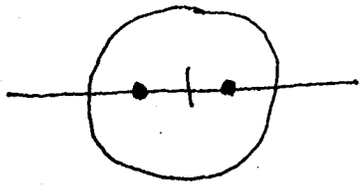
E-field due to the "negatively charged sphere":

$$\vec{E}_2 = -\frac{\rho r'}{3\varepsilon_0}\hat{r}'$$

Then: $$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{\rho}{3\varepsilon_0}(r\hat{r} - r'\hat{r}')$$

$$= \frac{\rho}{3\varepsilon_0} a\hat{a}$$, which is constant inside the cavity!

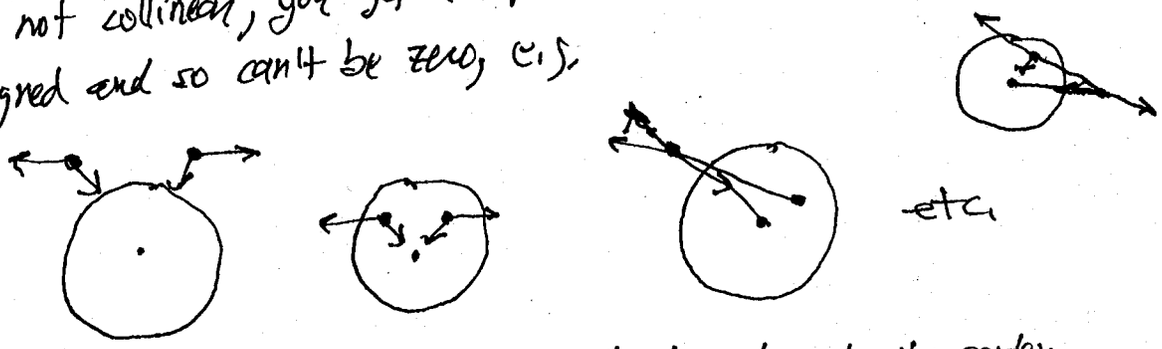
Problem 4



The answer was to place the protons symmetrically on both sides of the center of the electron jelly a distance $R/2$ from the center. The total force on each proton is then zero, giving mechanical equilibrium. (Remarkably, you should be able to show further this is a <u>stable</u> equilibrium in that small displacements of the protons from these location lead to forces that push the protons back to this location. So this is a useful model of H_2.)

Starting with a uniform spherical electron jelly of radius R and charge $-2e$, where can we place two protons so that the total force $\sum \vec{F}_i$ on each is zero?

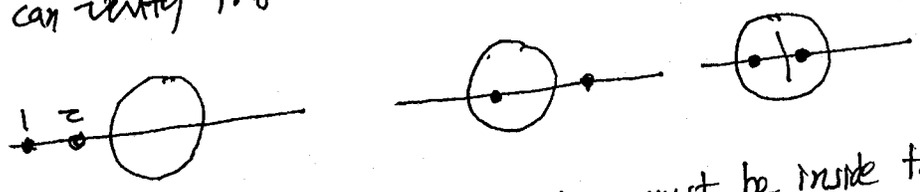
<u>A first deduction</u> all of you should have made is that the <u>protons must be collinear with the center of the jelly</u>. If they are not collinear, you get a repulsion and attraction that are not aligned and so can't be zero, e.g.



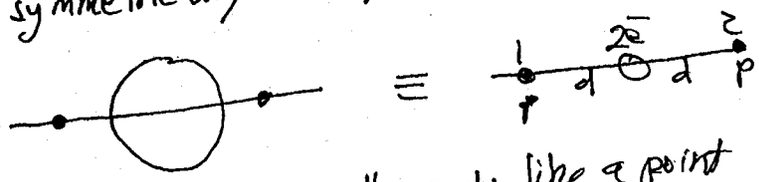
electron jelly produces attraction towards its center

A second deduction is that the protons must be symmetrically located on either side of the jelly's center. The reason is that each proton pushes the other proton away with a repulsive force of the same magnitude, Ke^2/d^2, where d is the distance between the protons. But if the protons are not symmetrically arranged with respect to the center, one proton will feel a different attractive force (weaker or stronger) than the other proton and so both can't both be in equilibrium, since they feel the same repulsion.

You can verify this with cases like these



A third deduction is that the protons must be inside the jelly. If they are outside and symmetrically arranged like this



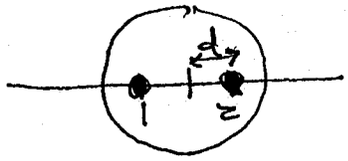
Then Gauss's law ~~tells~~ tells us that the jelly acts like a point charge $-2e$ at its center but now the forces can't balance for either proton because they are too far away

$$\text{attraction on } 1 = \frac{K(2e)e}{d^2} = \frac{2e^2K}{d^2}$$

$$\text{repulsion on } 1 = \frac{Ke(e)}{(2d)^2} = \frac{Ke^2}{4d^2}$$

attraction is 8 x stronger than repulsion for any d, won't work.

We conclude that only possible situation is two protons inside jelly, collinear with center, and symmetrically arranged like this:



Let d be distance of protons to center of jelly. Then proton 1 feels repulsion to left

$$\frac{Ke^2}{4d^2}$$

and we can find the attractive force on 1 to right by using Gauss's law applied to Gaussian bubble of radius d centered on sphere

$$(4\pi d^2)E = \frac{Q_{encl}}{\epsilon_0} = \frac{1}{\epsilon_0}\left[\left(\frac{\frac{4}{3}\pi d^3}{\frac{4}{3}\pi R^3}\right)(2e)\right] = \frac{2e}{\epsilon_0}\left(\frac{d}{R}\right)^3$$

or

$$E(d) = \frac{2eKd}{R^3} \quad \text{using} \quad K = \frac{1}{4\pi\epsilon_0}$$

To be in equilibrium, total force on 1 must be zero:

$$\frac{Ke^2}{4d^2} = \frac{2e^2Kd}{R^3} \Rightarrow d^3 = \frac{R^3}{8}$$

or $\boxed{d = R/2}$ as claimed on page 4

For the physical motivation for this jelly model, look up the "Plum pudding model" in Wikipedia, a 1904 model proposed by JJ Thompson to explain how stable atoms might be possible with static electrical forces. Thompson used positive jelly and electron "plums".

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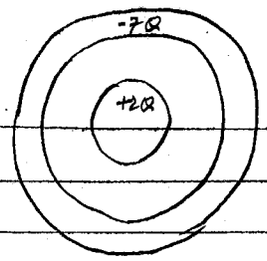

(a) We are considering conducting materials. So E-field inside the sphere and shell must be zero. We can structure a Gaussian surface as a sphere with radius larger than that of the inside of the shell and smaller than that of the outside of the shell.

$$\oint_S E_n \, dA = 0 = \frac{Q_{enclosed}}{\varepsilon_0}$$

and

$$Q_{enclosed} = +2Q + Q_{inner}$$

$$\Rightarrow \quad 0 = \frac{+2Q + Q_{inner}}{\varepsilon_0}$$

$$\Rightarrow Q_{inner} = -2Q.$$

By conservation of charge, we have:

$$Q_{outer} = -7Q - Q_{inner} = -5Q$$

(b) Because of the metal wire connecting the solid and shell, there is no E-field between them. Construct a spherical Gaussian surface inbetween the shell and the solid:

$$\oint_S E_n \, dA = 0 = \frac{Q_{enclosed}}{\varepsilon_0} \Rightarrow Q_{enclosed} = Q_{solid} = 0$$

$$\Rightarrow Q_{inner} = 0$$

$$Q_{outer} = -7Q + 2Q = -5Q$$

The E-field between the solid and the sphere changes from a finite value zero.

(c) The outer surface now has a zero potential, while the inner surface still has $-2Q$.

$$\Rightarrow \Phi = -\frac{1}{4\pi\varepsilon_0}(+2Q - 2Q + Q_{outer}) = 0 \Rightarrow Q_{outer} = 0$$

$$Q_{solid} = +2Q, \quad Q_{inner} = -2Q.$$

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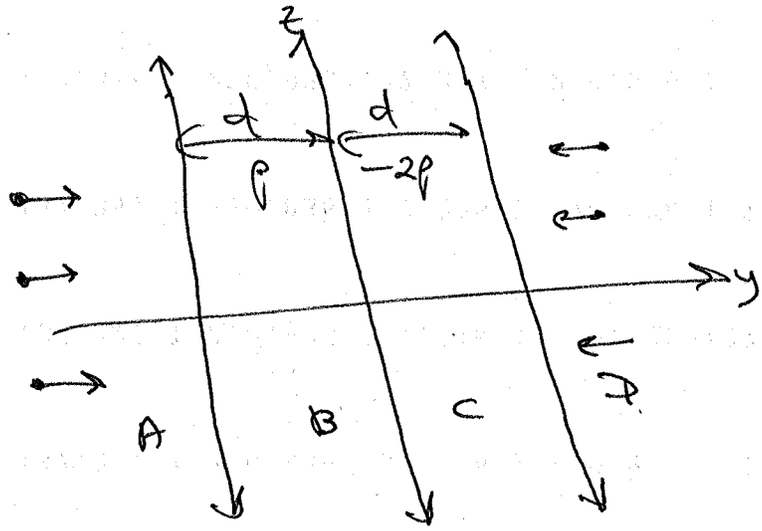
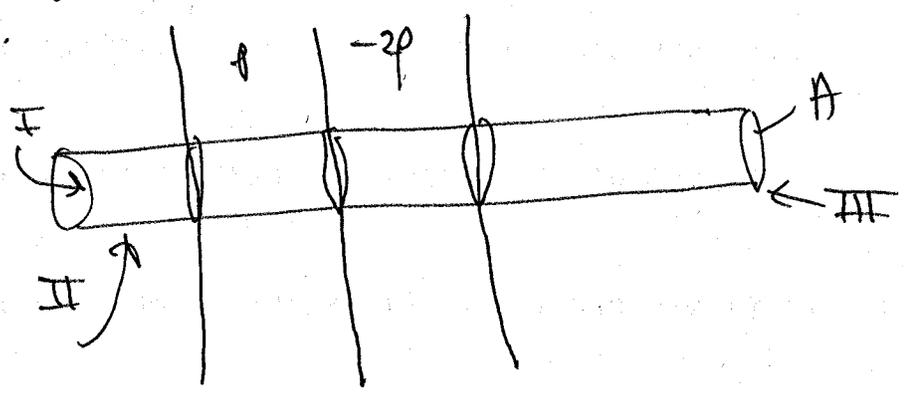
(9)

Problem 6a



Let's discuss two ways to solve this.

First is to observe that $\vec{E}$ in region A and $\vec{E}$ in region D must be equal and opposite in value, and uniform in A, uniform in D. So we can apply Gauss's law to Gaussian bubble spanning both slabs like this:



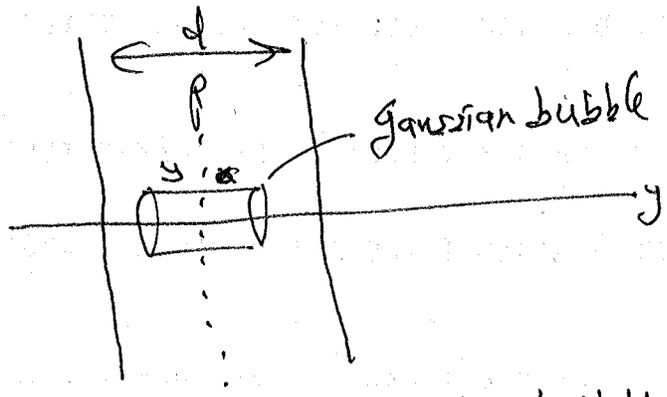
get $$\int \vec{E}\cdot d\vec{A} = \int_{I} \vec{E}\cdot d\vec{A} + \int_{II} \vec{E}\cdot d\vec{A} + \int_{III} \vec{E}\cdot d\vec{A} = -EA + 0 - EA$$

$$-2EA = \frac{Q_{encl}}{\epsilon_0} = \frac{1}{\epsilon_0}\left[(dA)(\rho) + (dA)(-2\rho)\right] = -\frac{A\rho d}{\epsilon_0}$$

$$\Rightarrow E = \frac{\rho d}{2\epsilon_0} \text{ is magnitude}$$

$$\vec{E}(y) = \begin{cases} \frac{\rho d}{2\epsilon_0}\hat{y} & y \leq -d \\ -\frac{\rho d}{2\epsilon_0}\hat{y} & y \geq d \end{cases}$$

To get $\vec{E}$ field inside left slab, region B where $-d \le y < 0$, apply Gauss's theorem to left slab <u>by itself</u> like this:



Gaussian is <u>symmetric</u> cylinder of area A, ends at $\pm y$. Gauss's law gives

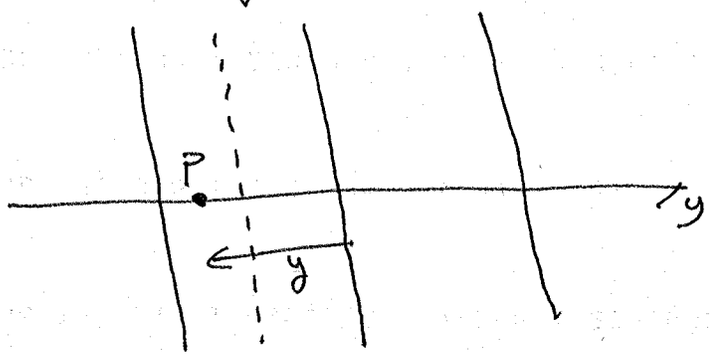
$$\int \vec{E}\cdot d\vec{A} = \underbrace{\int \vec{E}\cdot d\vec{A}}_{I} + \underbrace{\int \vec{E}\cdot d\vec{A}}_{II} + \underbrace{\int \vec{E}\cdot d\vec{A}}_{III} = EA + 0 + EA = 2EA$$

$$Q_{encl} = [(2y)A]\rho \Rightarrow 2EA = \frac{1}{\epsilon_0}[(2yA)\rho]$$

$$\boxed{E = \frac{|\rho| y}{\epsilon_0}}$$

magnitude of $\vec{E}$ inside slab at distance y from center of slab

Apply this to two slabs at point y s.t. $-d \le y < -\frac{d}{2}$



distance of P to midpoint of left slab is $\underbrace{-\frac{d}{2} - y}_{\text{positive since } -d<y<-\frac{d}{2}}$

so $\vec{E}$ field at point P at location $-y$ for $-d < -y < -\frac{d}{2}$ is:

$$\vec{E} = \underbrace{\underbrace{\frac{\rho\left(-\frac{d}{2}-y\right)}{\epsilon_0}}_{\text{magnitude from formula for uniform thick slab}}\underbrace{(-\hat{y})}_{\text{points to left since } \rho>0}}_{\vec{E}\text{ field from left slab with density } \rho} + \underbrace{\underbrace{\frac{|2\rho|\left(\frac{d}{2}\right)}{\epsilon_0}}_{\text{use same formula with } y=d/2}\underbrace{(\hat{y})}_{\text{points to right since } -2\rho<0}}$$

$$= \frac{\rho}{2\epsilon_0}(3d+2y)\,\hat{y} \qquad -d < y < -\frac{d}{2}$$

to right since $3d+2y>0$ if $-d<y<-\frac{d}{2}$

as given in the assignment, Eq. (5)

For $-\frac{d}{2} < y < 0$, get instead

$$\vec{E} = \rho\left[y+\frac{d}{2}\right](\hat{y}) + \underbrace{\frac{|2\rho|\left(\frac{d}{2}\right)}{\epsilon_0}\hat{y}}_{\text{doesn't change since still outside slab on right}}$$

sign changes $y-\left(-\frac{d}{2}\right) = y+\frac{d}{2}$

sign changes

$$= \frac{\rho}{2\epsilon_0}(3d+2y)\,\hat{y}$$ same expression

I'll let you work out the two cases

$$0 \le y \le \frac{d}{2}, \quad \frac{d}{2} < y < d$$

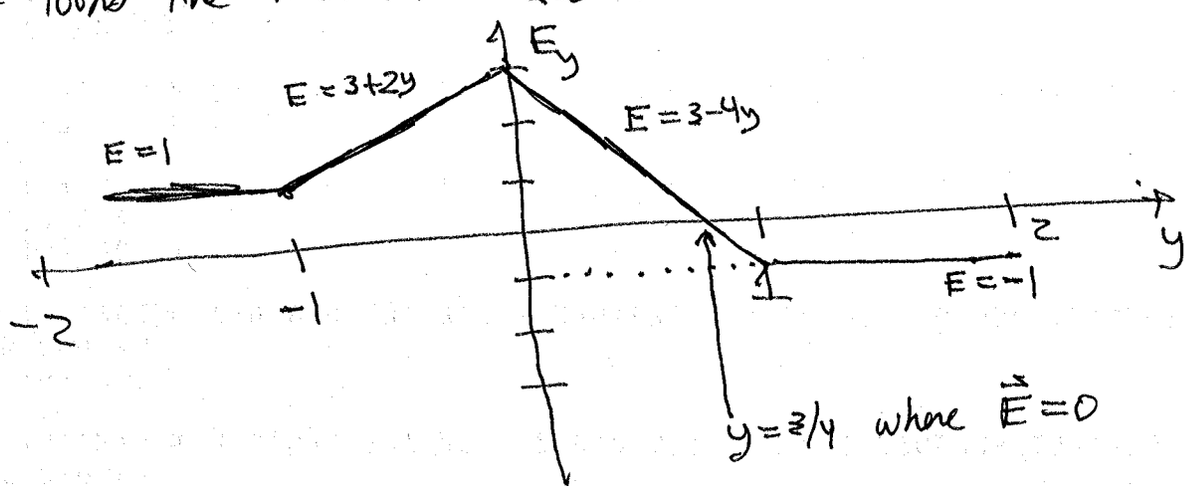
and verify the result

$$\vec{E} = \frac{\rho}{2\epsilon_0}(3d - 4y)\hat{y} \qquad 0 \le y \le d.$$

So entire solution is:

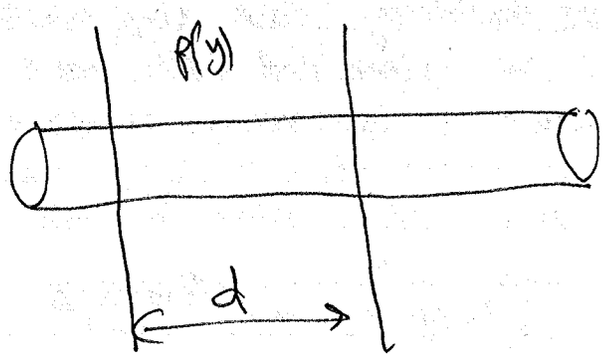
$$\vec{E}(y) = \begin{cases} \left(\frac{\rho}{2\epsilon_0}\right) d\,\hat{y} & y < -d \\ \left(\frac{\rho}{2\epsilon_0}\right)(3d + 2y)\,\hat{y} & -d \le y < 0 \\ \left(\frac{\rho}{2\epsilon_0}\right)(3d - 4y)\,\hat{y} & 0 \le y < d \\ -\left(\frac{\rho}{2\epsilon_0}\right) d\,\hat{y} & d \le y \end{cases}$$

Plot looks like this: for $\frac{\rho}{2\epsilon_0} = 1$ and $d = 1$:



$\vec{E}$ on left points to right, increases in strength to max at $y = 0$, then decreases to zero at $y = \frac{3}{4}$, then reverses direction to value -1 for $y \geq 1$

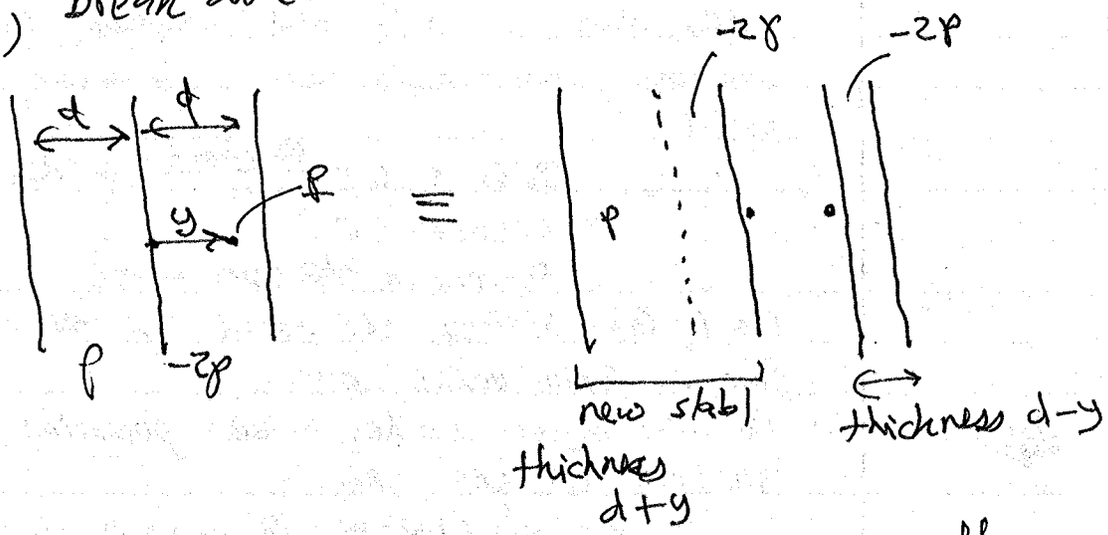
There is still another way to solve this problem, which is to observe you can apply Gauss's law to any slab of thickness d and arbitrary varying charge density $\rho(y)$:



$$2|E|A = \frac{Q_{encl}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_0^d \rho\,[A\,dy] = \frac{A}{\epsilon_0}\int_0^d \rho(y)\,dy$$

$$\Longrightarrow \boxed{|E| = \frac{1}{2\epsilon_0}\int_0^d \rho(y)\,dy}$$

So a tricky insight: at any point inside the double slab at coordinate y, break double slab into two new slabs like this:



and apply above formula twice. You will get the same result.

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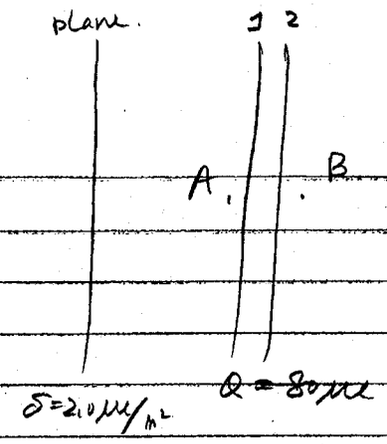

(a) The slab is conducting, so inside the slab E-field is zero. Suppose charge q_1, q_2 on the two sides of the slab:

$$E = E_1 + E_2 + E_{plane}$$

$$= \frac{q_1/A}{2\varepsilon_0} - \frac{q_2/A}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = 0$$

also:

$$q_1 + q_2 = 80\,\mu C$$

$$\Rightarrow \frac{(80\,\mu C - q_2)/25\,m^2}{2\varepsilon_0} - \frac{q_2/25\,m^2}{2\varepsilon_0} + \frac{2.0\,\mu C/m^2}{2\varepsilon_0} = 0$$

$$80\,\mu C - q_2 - q_2 + 50\,\mu C = 0$$

$$\Rightarrow q_2 = \frac{80\,\mu C + 50\,\mu C}{2} = 65\,\mu C$$

$$q_1 = 80\,\mu C - \mu_2 = 15\,\mu C$$

Below

(b) $E_{plane} = \frac{\sigma}{2\varepsilon_0} = \frac{2.0\,\mu C/m^2}{2\varepsilon_0} = \frac{1}{\varepsilon_0}\,\mu C/m^2$

$$F = E_{plane}(q_1 + q_2) = \frac{1}{\varepsilon_0}\,\mu C/m^2 \cdot 80\,\mu C \approx 9.0\,N$$

$$P = \frac{F}{25\,m^2} = 0.36\,N/m^2 = 3.6 \times 10^{-6}\,atm.$$

a) continued:

$$\vec{E}_A = \left(\frac{\sigma}{2\varepsilon_0} - \frac{q_1/A}{2\varepsilon_0} - \frac{q_2/A}{2\varepsilon_0}\right)\hat{y} = -7 \times 10^4\,\hat{y}\,N/C$$

$$\vec{E}_B = \left(\frac{\sigma}{2\varepsilon_0} + \frac{q_1/A}{2\varepsilon_0} + \frac{q_2/A}{2\varepsilon_0}\right)\hat{y} = 2.9 \times 10^5\,\hat{y}\,N/C$$