1. The following problem provides some practice and review with the vector product \( \mathbf{a} \times \mathbf{b} \) of two vectors. Pages 337-339 of the Knight text also would be worth reviewing.

(a) What is the value of
\[
\hat{i} \times \left[ (\hat{k} \times \hat{i}) \times (\hat{j} \times \hat{k}) \right]
\]
(1)

Note: you should be able to do this in your head, without using coordinates or any substantial calculation, by using the fact that the three vectors \( \hat{i}, \hat{j}, \) and \( \hat{k} \) geometrically define a right-handed Cartesian coordinate system.

(b) Show that the vector product \( \times \) of two vectors is not associative by finding three vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) such that
\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}.
\]
(You already know that the cross product is not commutative since \( \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \).)

(c) Using the coordinate representation of the cross product of two vectors
\[
\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y) \hat{x} + (a_z b_x - a_x b_z) \hat{y} + (a_x b_y - a_y b_x) \hat{z},
\]
(2)
where
\[
\mathbf{a} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \quad \text{and} \quad \mathbf{b} = b_x \hat{x} + b_y \hat{y} + b_z \hat{z},
\]
(3)
show that
\[
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0,
\]
(4)
that is the dot-product of the vector \( \mathbf{a} \times \mathbf{b} \) with either vector \( \mathbf{a} \) or vector \( \mathbf{b} \) is zero. This proves using coordinates that the cross-product of two vectors is perpendicular to both vectors, which better be the case because the cross-product of two vectors is defined to be perpendicular to the plane of the two vectors via a right-hand rule.

(d) Consider two vectors
\[
\mathbf{v}_1 = \hat{i} - 3\hat{j} + 2\hat{k},
\]
(5)
\[
\mathbf{v}_2 = 5\hat{i} - \hat{j} - \hat{k}.
\]
(6)
Use Eq. (4) in a constructive way to find the unit vector \( \hat{n} \) that is perpendicular to \( \mathbf{v}_1 \), perpendicular to \( \mathbf{v}_2 \) and that has a positive \( z \) component. This unit vector is one way to indicate the “direction” of the plane that contains the two vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \).

Also find the angle \( \theta \) between vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) in degrees to two significant digits. You can do this in two ways, one is to use \( \mathbf{a} \cdot \mathbf{b} = ab\cos(\theta) \), or use \( |\mathbf{a} \times \mathbf{b}| = ab\sin(\theta) \).

(e) If \( \mathbf{a} \neq \mathbf{0} \) is a non-zero three-dimensional vector, explain briefly why \( \mathbf{a} \cdot \mathbf{b} = 0 \) does not by itself imply \( \mathbf{b} = \mathbf{0} \) and similarly explain why \( \mathbf{a} \times \mathbf{b} = \mathbf{0} \) by itself does not imply \( \mathbf{b} = \mathbf{0} \). But if \( \mathbf{a} \cdot \mathbf{b} = 0 \) and \( \mathbf{a} \times \mathbf{b} = \mathbf{0} \) for some \( \mathbf{a} \neq \mathbf{0} \), does \( \mathbf{b} = \mathbf{0} \)?

2. At some particular time, two identical point particles each with charge \( q > 0 \) are located at the coordinates \( \mathbf{r}_1 = (0, 0, 0) \) and \( \mathbf{r}_2 = (0, y_0, 0) \). They are both moving with velocity \( \mathbf{v} = v\hat{x} \) in the positive \( x \) direction, where the speed \( v \) is much smaller than the speed of light \( c \) (an assumption...
needed for the Biot-Savart law to be valid). Show that the ratio $F_B/F_E$ of the magnitude of the magnetic force $F_B$ to the magnitude of the electric force $F_E$ on each particle is given by

$$
\frac{F_B}{F_E} = \epsilon_0 \mu_0 v^2.
$$

(7)

3. For the current carrying loop in this figure,

![Diagram of a current-carrying loop](image)

show that the magnetic field vector $\mathbf{B}$ at the point $P$ has the magnitude

$$
B = \frac{\mu_0 I}{4} \left( \frac{1}{R_1} - \frac{1}{R_2} \right),
$$

(8)

and give the direction of $\mathbf{B}$. Also determine the magnetic moment $\mu$ of this loop in terms of the given parameters.

4. Consider this coaxial cable

![Diagram of a coaxial cable](image)

which consists of a solid inner cylindrical conductor of radius $R_1$ surrounded by a hollow shell of negligible thickness of radius $R_2$. A current $I$ flows through both conductors but in opposite directions, and the current density is uniformly distributed over the solid conductor.

(a) Use Ampère’s law to find expressions for the magnetic field $\mathbf{B}$ in three regions: within the inner conductor, in the space between the conductors, and outside the outer conductor.

(b) Draw a graph of magnetic field magnitude $B(r)$ versus radius $r$ from $r = 0$ to $r = 2R_2$ for the case $R_1 = (1/3)R_2$.

5. A current $I$ flows through solenoid 1 which has a radius $R_1 = R$ and $n_1$ coils per meter. Solenoid 1 is placed inside and coaxial with a second solenoid, solenoid 2, which has a radius $R_2 = 2R$ and winding density $n_2 > n_1$ but the current $I$ in solenoid 2 moves in the direction opposite to the current in solenoid 1. (So looking down the axis of the two solenoids will look like two concentric circles of radius $R_1$ and $R_2$ with currents going in opposite directions.) Both solenoids are extremely long and can be considered ideal to a good approximation. Derive equations for the magnitude of the $\mathbf{B}$ field everywhere in space, and explain what are the directions of the $\mathbf{B}$ field at distances $r = 0$, $r = 1.5R$, and $r = 3R$ from the common axis. (Hint: you do not need Ampère’s law here, just use the known properties of an ideal solenoid and superposition.)

6. Three long parallel straight wires pass through the vertices of an equilateral triangle with side $L = 10$ cm as shown here:
The dot in the upper circle indicates current flowing out of the page, while the two X’s in the lower circles denote current flowing into the page. If each current is 15 A, find

(a) the magnetic field \( \mathbf{B} \) at the location of the upper wire.
(b) the force per unit length on the upper wire.

Answers to one significant digit: (a) 50 \( \mu \)T to right; (b) \( 8 \times 10^{-4} \) N/m up.

7. A nonuniform (inhomogeneous) magnetic field exerts a net force on a magnetic moment. To see this, consider a circular current loop of radius \( R \) that is placed near the diverging magnetic field of the north pole of a bar magnet as shown here:

![Diagram of a circular current loop near a bar magnet](image)

By symmetry, the magnetic field makes the same angle \( \theta \) with respect to the vertical at any point on the ring.

(a) Show that the net magnetic force \( \mathbf{F} \) on the ring is \( 2\pi RI B \sin(\theta) \).
(b) Estimate the force on the ring to the nearest power of ten if \( R = 2 \) cm, \( I = 0.5 \) A, \( B = 200 \) mT, and \( \theta = 20^\circ \). (Answer to one digit: \( 4 \times 10^{-3} \) N but give your answer to two digits.)

8. A proton with speed \( 1.00 \times 10^6 \) m/s enters a region with a uniform magnetic field that has a magnitude \( B = 0.80 \) T and direction into the page as shown here

![Diagram of a proton entering a magnetic field](image)

The proton enters the region with an angle \( \theta = 60^\circ \). Determine the exit angle \( \phi \) and show that the distance \( d \approx 13 \) mm (but you should give your answer to three digits).
9. A rectangular current-carrying 50-turn coil as shown here pivots about the $z$-axis.

(a) If the wires in the $xy$ plane make an angle $\theta = 37^\circ$ with the $y$-axis, what angle does the magnetic moment of the coil make with the unit vector $\hat{i}$?

(b) In terms of the unit vectors $\hat{i}$ and $\hat{j}$, write an expression for the unit vector $\hat{n}$ that points in the direction of the magnetic moment $\mu$.

(c) What is the magnetic moment vector $\mu$ of the coil?

(d) Find the torque $\tau$ on the coil when the coil is immersed in a uniform magnetic field given by $\mathbf{B} = (1.5 \text{ T})\hat{j}$.

(e) Find the potential energy $U = -\mu \cdot \mathbf{B}$ of the coil in this magnetic field.

Some partial answers: (b) $\hat{n} \approx 0.8\hat{i} - 0.6\hat{j}$, (c) $\mu \approx (0.28 \text{ A} \cdot \text{m}^2) \hat{i} - (0.21 \text{ A} \cdot \text{m}^2) \hat{j}$, (d) $\tau \approx (0.42 \text{ N} \cdot \text{m}) \hat{k}$, and (e) $U \approx 0.32 \text{ J}$.

10. The Earth is a negatively charged rotating sphere with radius $R \approx 6,400 \text{ km}$, rotation rate of once every 24 hours, and with an approximately uniform surface charge density corresponding to about $10^{10}$ electrons per square meter. From these data, estimate to the nearest power of ten the ratio $B_N/B_0$ where $B_N$ is the magnitude of the magnetic field at the North pole caused by the Earth’s rotating surface charge, and where $B_0 \approx 10^{-5} \text{ T}$ is the experimentally known magnitude of the Earth’s surface magnetic field. Your estimate answers the interesting question of whether there is a sizable contribution to the Earth’s surface magnetic field from its rotating surface charge.

Note: Example 32.5 on page 930 of the Knight book will be helpful.

11. Time to Complete This Assignment

To one significant digit, please give the time in hours that it took you to complete this homework assignment.

12. Optional Challenge Problems

(a) Challenge problem 8.1: in class, you will learn how to use Ampère’s law to deduce the magnetic field magnitude $B(r)$ at points inside a toroid (a donut-shaped solenoid that has been bent into a closed ring, the key geometry of a fusion tokamak) that are a distance $r$ from the axis of the toroid. The argument requires a fact that I give in class without derivation, which is that at any point inside the toroid, the magnetic field vector $\mathbf{B}$ is tangential to the circle centered on the axis and passing through the point (and consistent with the right-hand rule); it is this assumption that makes it easy to calculate the line integral $\oint \mathbf{B} \cdot d\mathbf{l} = B(2\pi r)$ involved with Ampère’s law. For this challenge problem, apply the Biot-Savart law to the various loops winding around the toroid
to show that, at any point inside the toroid, the magnetic fields due to the many loops combine in such a way that the resulting $\mathbf{B}$ field at any point inside the toroid is indeed tangential to a circle passing through that point that is centered on the axis of the toroid.

(b) Challenge problem 8.2: for fun, here are a few miscellaneous math problems to challenge and so improve your spatial reasoning:

i. Assuming that the earth is a sphere, where on the earth’s surface is it possible for a person to walk one kilometer south, one kilometer east, and one kilometer north and end up in the exact same place? Note: there is more than one such place.

ii. What is the maximum number of distinct regions that five planes can divide space into?

iii. Explain how to arrange five points A, B, C, D, and E in three-dimensional space so that they have these properties:
   A. $AB = BC = CD = DE = EA = 1$
   B. $\angle ABC = \angle CDE = \angle DEA = 90^\circ$.
   C. Segment DE is parallel to the plane of triangle ABC.

iv. What is the largest possible radius of a circle that is contained within a 4-dimensional hypercube whose side has length 1?