

Physics 162 Assignment 7

Made available: Wednesday, February 25, 2015

Due: Monday, March 16, 2015 by 9 pm.

1. **Limits of formula for electric field on axis of a charged cylindrical shell** This is a review problem, to continue to improve your skills in using the generalized binomial theorem.

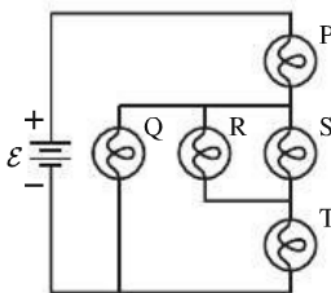
One can show that the z -component $E_z(z)$ of the electric field on the axis produced by a cylindrical shell of radius R , length L , and surface charge density σ is given by

$$E_z(z) = 2\pi K\sigma \left(\left[1 + \left(\frac{z - L/2}{R} \right)^2 \right]^{-1/2} - \left[1 + \left(\frac{z + L/2}{R} \right)^2 \right]^{-1/2} \right), \quad (1)$$

where the axis of the shell is the z -axis and $z = 0$ corresponds to the center of the shell. Use the generalized binomial theorem to show that Eq (1):

- (a) reduces correctly to the electric field KQ/z^2 of a point particle with appropriate charge Q when z corresponds to a point on the axis far from the shell, that is for $z \gg L, R$.
- (b) reduces correctly to the electric field $KQz/(z^2 + R^2)^{3/2}$ of a point on the axis of a ring of radius R with appropriate total charge Q when the length of the solenoid $L \ll R$ is small compared to the radius of the ring, i.e., short solenoids act like rings.

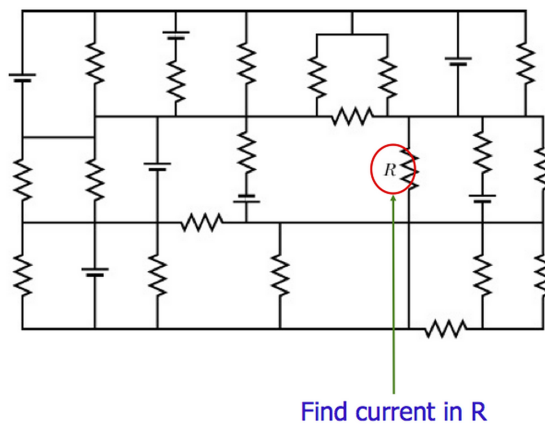
2. **Relative brightness of identical lightbulbs** The five identical bulbs P, Q, R, S, and T in this figure



are all glowing. Deduce and give the order of brightness of the bulbs, from brightest to dimmest.

Note: make the simplifying assumption that the resistance of each bulb is constant (does not vary with the brightness of the bulb). This is not a good assumption for filament-based (incandescent) light bulbs for which the filament resistance does increase with brightness, which corresponds to higher filament temperature. In the Modern Physics or Thermal Physics courses at Duke (264 or 363), you will learn an interesting fact that the light emitted by filaments is accurately understood as so-called “black-body radiation” and has a universal spectrum (light intensity versus frequency) that depends just on the temperature of the filament.

3. **A crazy circuit with a simple insight** The following circuit has identical batteries all with emf $\mathcal{E}$ and identical resistors all with resistance R .

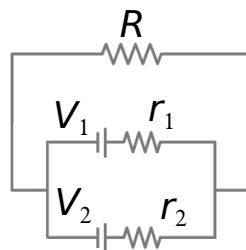


By applying Kirchhoff's loop law to a carefully chosen loop, determine in just a few seconds and without any algebra the magnitude and direction of the current I flowing through the circled resistor R .

4. **Maximizing how long a light source will last** You are lost in a dark underground cave system, but have in your spelunker's backpack numerous wires and two identical flashlights that each contain a bulb of resistance R and an ideal battery (zero internal resistance) of emf $\mathcal{E}$.
- To maximize your chance of finding a way out of the cave system, explain with a circuit diagram how to combine some or all of these items (wires, bulbs, batteries) to make a light source (a single circuit that you can not change after assembling it) that will last as long as possible. How much longer will your circuit last compared to using each flashlight separately in succession, until each runs out of energy?
 - If the flashlight bulbs are the classical 1.4-watt 2.4-volt incandescent flashlight bulbs (a dying technology since new flashlights use LED-based light sources which are more power efficient), and the batteries are 1.5-volt alkaline D-cells, each with an energy capacity of about 20 watt-hours (Wh), estimate to the nearest power of ten how long in hours your circuit will produce light. (For simplicity, assume that the emf of the batteries is constant until the batteries run out of energy, not a good assumption.)

Note: you may enjoy watching the 2005 British horror movie "The Descent" about six women who enter an unmapped cave system in North Carolina. And all of you should play the classic text-based rather hard but pioneering adventure game "Colossal Cave" (there are free copies scattered around the Internet for all computer systems) and see if you can figure out how to get out of its famous maze of many twisty little passages.

5. Two batteries that have emfs of V_1 and V_2 and internal resistances of r_1 and r_2 are connected in parallel, positive terminal to positive terminal, negative terminal to negative terminal. If a resistor of resistance R is then connected in parallel with these batteries like this:

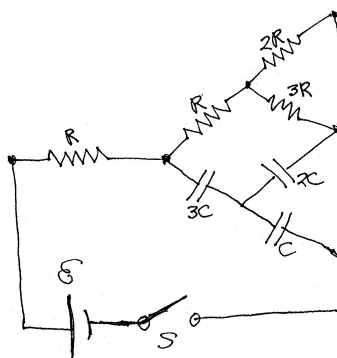


show that the value of R for which the two batteries produce the most power is given by

$$R = \frac{r_1 r_2}{r_1 + r_2}. \quad (2)$$

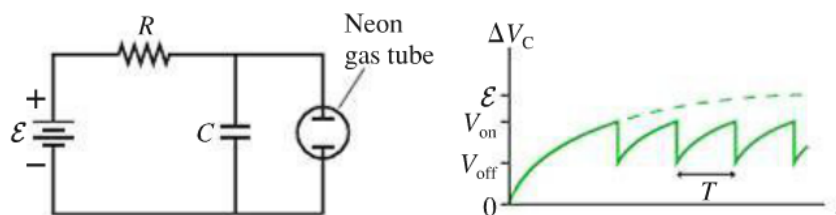
Hint: This is a two-loop circuit with two junctions so use Kirchhoff's rules to obtain three linear equations in three unknown currents, solve for the current passing through the resistor R , and then maximize the power being dissipated by the resistor. Also make sure to show that your answer gives a maximum, not just an extremum.

6. Consider the following circuit consisting of a battery with emf $\mathcal{E}$, an open switch S , and an array of resistors and discharged capacitors with resistances and capacitances as indicated:



- (a) Immediately after the switch S is closed and in terms of the quantities $\mathcal{E}$, R , and C :
- what is the power P being produced by the battery?
 - what is the current I_{2R} through the resistor of resistance $2R$?
 - what is the current I_C through the capacitor with capacitance C ?
- (b) A long time after the switch S has been closed and in terms of the quantities $\mathcal{E}$, R , and C :
- what is the power P being produced by the battery?
 - what is the current I_{2R} through the resistor of resistance $2R$?
 - what is the current I_C through the capacitor with capacitance C ?
 - what is the magnitude $|Q_C|$ of the charge on the capacitor with capacitance C ?
- (c) A long time after the switch S has been closed, the switch is opened. If $\mathcal{E} = 100 \text{ V}$, $R = 1 \text{ M}\Omega$, and $C = 1 \mu\text{F}$
- To the nearest power of ten, what is the total energy U in joules stored in the circuit immediately after the switch is opened?
 - To one significant digit, how long (in seconds) will it take for this energy to decrease to one tenth of this initial value?

7. **An electronic relaxation oscillator and simple model of neuronal action potentials** As shown and discussed in class, an oscillator circuit can be built by adding a neon gas tube to an RC -circuit like this



Gas is normally a good insulator and the resistance of the gas tube is effectively infinite when the light is off, which allows the capacitor to charge. But when the potential difference across the capacitor reaches a threshold V_{on} , the electric field inside the gas becomes strong enough to ionize the neon gas (the so-called dielectric strength of the gas is exceeded), the resistance becomes small, a current flows, the bulb lights with an orange glow, and the capacitor discharges rapidly through the bulb. When the capacitor voltage has decreased below a value V_{off} , the electric field inside the gas becomes too weak to sustain the ionization, the gas becomes a poorly conducting insulator again, the light turns off, and the capacitor starts to charge as shown in the right panel of the figure. The capacitor voltage $V(t)$ then oscillates periodically (but not sinusoidally!) between the values V_{off} and V_{on} .

- (a) Show that the oscillation period T of this circuit is

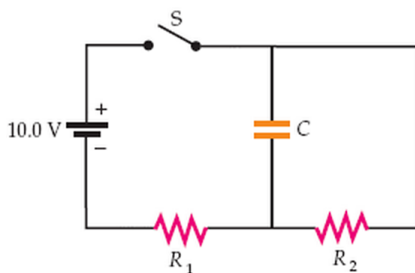
$$T = RC \ln \left(\frac{\mathcal{E} - V_{\text{off}}}{\mathcal{E} - V_{\text{on}}} \right). \quad (3)$$

- (b) A neon gas tube has $V_{\text{off}} = 20 \text{ V}$ and $V_{\text{on}} = 80 \text{ V}$. What resistance R should be used in combination with a $10 \mu\text{F}$ capacitor and a 90 V battery to create a 10 Hz oscillator?

This problem is interesting for another reason, it mimics rather well one of the simplest models of how neurons in the brain generate their digital pulses called action potentials. (The simple mathematical for a neuron is called a “leaky integrate and fire model”.) A neuron is like an RC -circuit where C is the capacitance of the neuron (cell body) and R is the resistance across the lipid bilayer. When a neuron is stimulated, certain channels (complicated proteins in the lipid bilayer) change shape and create holes through the membrane allowing some current to flow and this causes the voltage difference across the lipid bilayer to start increasing from about -80 mV towards zero, just like the voltage across the capacitor climbing in the neon oscillator circuit. And when the potential difference across the neuronal membrane reaches a threshold analogous to V_{on} , new channels open causing a fast large current to flow, causing temporarily a large increase in the potential difference (to $+30 \text{ mV}$) before the potential difference drops back to -80 mV . A neuron that has a constant current stimulus will undergo oscillations whose period is given by a formula that is essentially the same as Eq. (3).

8. **The next more complicated RC circuit** If a circuit has capacitances and resistances that are not connected in series with a battery, one has little choice but to work out the details specifically for each problem, deriving differential equations for each current that determine how the currents evolve over time. The following is a guided tutorial to work through the details of a circuit that is just a little more complicated than a series RC circuit. This tutorial should also help you become more comfortable manipulating differential equations, an important skill in many areas of science and engineering.

Consider the circuit below consisting of a battery with emf $\mathcal{E}$, two resistors with resistances R_1 and R_2 , and a capacitor with capacitance C :



You can think of this circuit as a capacitor C in parallel with a resistance R_2 that are then connected in series to a second resistance R_1 (with R_1 possibly including the internal resistance r of the battery). Initially, the switch has been open for a long time so that the capacitor C has fully discharged through the resistance R_2 , and there are no currents flowing anywhere in the circuit. At time $t = 0$, the switch S is closed and then, at time $t = 2$ s later, the switch is opened. Your goal is to understand qualitatively and quantitatively how the current $I_2(t)$ through the resistance R_2 varies with time for $t \geq 0$.

- (a) Explain briefly how the circuit in the above figure represents the same problem as that of Problem 6 above. So solving the current problem will tell you how the currents and charges on the capacitors in Problem 6 behave at all moments in time, not just at early and late times.
- (b) Before deriving and studying the differential equations that describe how the currents and capacitor charge evolve over time, understand the key points conceptually by deducing and giving the short-time and long-time values of:
 - i. the current $I_1(t)$ flowing through the resistance R_1 from right to left;
 - ii. the current $I_2(t)$ flowing through the resistance R_2 from right to left;
 - iii. the current $I_3(t)$ flowing downwards through the capacitor;
 - iv. the charge $Q(t)$ on the positive plate of the capacitor.

Here “short-time” means immediately after the switch S has been closed, and “long-time” means a long time after the switch S has been closed.

- (c) In preparation of applying Kirchhoff’s circuit laws for $t \geq 0$, introduce three currents associated with the junction at the top of the capacitor: a current I_1 that goes clockwise (CW) through R_1 through the battery, and through the switch; a current I_2 that passes through the resistance R_2 also going CW, and a current I_3 that flows downwards through the capacitance C .

Now apply Kirchhoff’s laws to obtain three equations for the unknown currents $I_1(t)$, $I_2(t)$, and $I_3(t)$ and also write down a fourth equation that relates the charge $Q(t)$ on the positive plate of the capacitor to the current I_3 . You will then have four equations in the four unknowns I_1 , I_2 , I_3 , and Q .

- (d) Use your four equations in part (c) to show that the charge $Q(t)$ on the capacitor evolves in time according to this first-order differential equation

$$\frac{dQ}{dt} = c_1 + c_2 Q, \quad \text{with} \quad c_1 = \frac{\mathcal{E}}{R_1} \quad \text{and} \quad c_2 = -\frac{1}{C} \left(\frac{1}{R_1} + \frac{1}{R_2} \right). \quad (4)$$

- (e) Use your equations from part (c) to show that

$$I_2(t) = \frac{1}{CR_2} Q, \quad (5)$$

and so the current $I_2(t)$ evolves in time exactly as the charge $Q(t)$, up to a constant factor.

- (f) Show by substitution of the expression

$$Q(t) = a + be^{-t/\tau}, \quad (6)$$

into Eq. (4) that Eq. (6) is a solution of the differential equation Eq. (4) that satisfies the initial data

$$Q = 0 \quad \text{at } t = 0, \quad (7)$$

provided that the constants a , b , and time constant τ have the values

$$a = \frac{R_2}{R_1 + R_2} \mathcal{E}C, \quad b = -a, \quad \tau = C \frac{R_1 R_2}{R_1 + R_2}. \quad (8)$$

So the charge Q on the capacitor explicitly evolves in time according to the expression

$$Q(t) = \frac{R_2}{R_1 + R_2} \mathcal{E}C \left(1 - e^{-t/\tau}\right), \quad (9)$$

and this is also how the current $I_2 = Q/(CR_2)$ through resistance R_2 evolves over time, starting from a discharged capacitor.

Draw qualitatively how Eq. (9) varies with time for $t \geq 0$ and discuss briefly whether Eq. (9) is consistent with your short- and long-time conclusions of part (b) above.

- (g) But Eq. (9) is *not* how the current $I_3(t)$ through the branch containing the capacitor evolves in time. Give an explicit mathematical expression for I_3 and plot $I_3(t)$ qualitatively for $t \geq 0$. Discuss briefly whether your plot is qualitatively consistent with your short- and long-time conclusions from part (b) above.
- (h) Give an explicit expression for the current $I_1(t)$ through the resistance R_1 as a function of time for $t \geq 0$ and draw a qualitative plot of I_1 . Discuss briefly whether your plot is consistent with the short- and long-time behavior of part (b) above.
- You now have a complete explicit knowledge of how all the currents and charge Q evolve in time, starting from when the switch is first closed.
- (i) Explain mathematically and separately physically why, in the limit that the resistance R_2 becomes large compared to R_1 , $R_2 \gg R_1$, the time constant τ in Eq. (8) reduces to the time constant CR_1 for the simpler series RC series circuit discussed in class and in the Knight book.
- Note: the time constant τ in Eq. (8) should be somewhat surprising to you since it looks like the time constant of a series RC circuit with resistance $R = 1/(1/R_1 + 1/R_2)$ of two resistances R_1 and R_2 in parallel, but the two resistances in this problem are not in parallel.
- (j) Finally get some specific physical insight by using the numerical parameters:

$$\mathcal{E} = 10 \text{ V}, \quad R_1 = 2.0 \text{ M}\Omega, \quad R_2 = 5.0 \text{ M}\Omega, \quad C = 1.0 \text{ }\mu\text{F}, \quad (10)$$

to sketch the graph qualitatively correctly of the current $I_2(t)$ versus time over the time interval $0 \leq t \leq 10$ if the switch is closed at $t = 0$ and opened at $t = 2$ s. Give also the explicit values of $I_2(t)$ at time $t = 2$ s and at time $t = 8$ s.

Warning: the time dependence of $I_2(t)$ after $t = 2$ s is different from Eq. (6)!

Note: all the above expressions are simple enough that you should be able to sketch their qualitative behavior directly, but you are also welcome to generate plots using Mathematica or some other graphing software. For example, in Mathematica, you can plot the expression Eq. (6) over the time interval $0 \leq t \leq 5$ by typing the following:

```
a = 2 ;
b = -2 ;
tau = 1 ;
```

```

Plot[
  a + b Exp[-t/tau] ,
  {t, 0, 5} ,
  PlotRange -> All
]

```

and then hitting Shift-Return to execute these statements. Mathematica can also solve many ordinary differential equations symbolically, and more successfully and accurately than most mathematical experts. For example, you can solve Eq. (4) symbolically by typing

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DSolve[ Q'[t] == c1 + c2 Q[t] , Q[t], t ]
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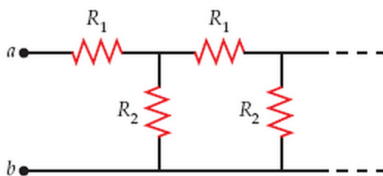
then by typing Shift-Return to execute this statement. “DSolve” is an intrinsic Mathematica function that stands for “Differential equation Solver”.

9. Time to Complete This Assignment

To one significant digit, please give the time in hours that it took you to complete this homework assignment.

10. Optional Challenge Problems

- (a) Challenge problem 7.1: An experimental and practical puzzle: you are given a complex electrical circuit consisting of many wires, resistors, and capacitors all soldered together and you wish to determine the resistance of a particular resistor in the circuit without cutting or unsoldering any wires. Assuming that you have a battery of known voltage V , an ammeter, a voltmeter, and as many wires as you need of negligible resistance, explain how to measure the resistance of some resistor in the middle of some circuit.
- (b) Challenge problem 7.2: what is the resistance between a and b of the following infinite circuit of pairs of resistances R_1 and R_2 ?



- (c) Challenge problem 7.3: This is one of the simplest problems related to polarization of a metal object by a point charge one can solve analytically. If you take the time to solve the problem and some plotting program to explore various features of the solution, you will get a lot of useful insight about actual polarization of a metal sphere by a point charge.

Consider a mathematical spherical surface of radius R located at the origin of some xyz -coordinate system and consider a point charge q located a distance $d > R$ from the center of this surface on the x -axis. Show that it is possible to place a point charge Q (whose value Q you need to determine) at some location on the x axis (which you also need to determine) such that the total potential V due to the charge Q and charge q is zero everywhere over the surface of the sphere.

Your answer provides a solution to two interesting practical problems: given a conducting sphere of radius R that has been grounded so that its surface is an equipotential surface with $V = 0$, and given a point charge q located outside the conducting sphere, what is the force acting on the charge q ? Alternatively, given a charge Q placed inside a conducting spherical shell at a point other than the center, what is the force acting on Q ? Your answer solves this problem because

there is a mathematical uniqueness theorem which says that, for any configuration of charges and conductors, there is only one possible potential $V(x, y, z)$ that is equipotential on the surfaces of the conductors. Since your two point charges produce an equipotential surface with $V = 0$ on the sphere, the total electric field $\mathbf{E}_{\text{total}}$ of the two points outside the sphere must be the same electric field produced by the charge q outside of a conducting sphere of radius R .

Finish this problem by using your two-charge solution to calculate the force acting on the charge q . Does your solution make sense in the limit that q is close to the surface of the sphere and in the limit when q is far from the center of the sphere?

Can you further figure out how to calculate the surface charge density σ at any point on the sphere? Can you figure out how to extend your solution to an isolated (not-grounded) conducting sphere with a net charge q' ?