Physics 162 Assignment 5

Made available: Sunday, February 8, 2014
Due: Sunday, February 15, 2015, by 10 AM.

Problems

1. In some region of space, the electric potential $V(x, y)$ is given by

$$V(x, y) = 100 \left(x^2 - y^2\right) \text{ V}, \tag{1}$$

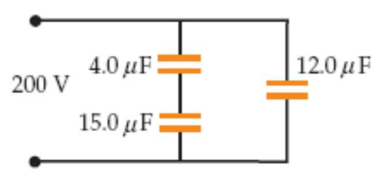
where the spatial coordinates x and y have units of meters. (Note the subtle choice of fonts: math italic V indicates the electric potential while the non-italic font V denotes the volt unit of the potential.)

   (a) Draw an approximate contour map of the potential by showing and labeling the equipotential surfaces corresponding to the five potential values $-400\,\text{V}$, $-100\,\text{V}$, $0\,\text{V}$, $100\,\text{V}$, and $400\,\text{V}$

   (b) Find an expression for the electric field vector $\mathbf{E}(x, y)$ at the general point (x, y).

   (c) Draw qualitatively the electric field lines on your contour map of part (a), using the fact that field lines are perpendicular to equipotential surfaces.

2. For this circuit



   (a) find the equivalent capacitance C_{eff}. (The answer to two significant digits is $15\,\mu\text{F}$, but give your answer to three digits.)

   (b) find the three charges Q_i on the positive plates of the three capacitors. (As a check, the charge on the $4\,\mu\text{F}$ capacitor is $\approx 0.6\,\text{mC}$ to one digit but give your answer to two digits.)

   (c) find the voltage difference ΔV_i across each capacitor. (Answers to two digits: 200 V, 160 V, and 42 V but give your answer to three digits.)

   (d) find the energy stored in each capacitor. (As a check, the energy stored in the $12\,\mu F$ capacitor is $240\,\text{mJ}$.)

3. A 20-pF capacitor is charged to $3.0\,\text{kV}$, removed from the battery, and then connected to an uncharged 50-pF capacitor.

   (a) What is the new charge on each capacitor? (Answers: to one digit, $20\,\text{nC}$ and $40\,\text{nC}$.)

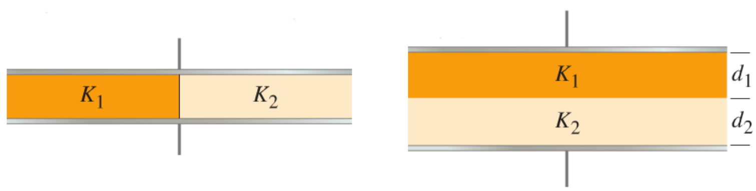
   (b) Find the energy stored in the 20-pf capacitor before it is disconnected from the battery, and the energy stored in the two capacitors after they are connected to each other. Does the stored energy increase or decrease by connecting the two capacitors to each other? Explain why energy is not conserved.

Hint: when you connect the two capacitors to each other, the connected plates of each capacitor form two equipotential surfaces, which implies that the voltage difference across the two capacitors must be the same in electrostatic equilibrium.

4. Demonstrate that you are a capacitance guru by showing that there are 23 distinct capacitances that can be obtained by connecting together, in different ways, five identical capacitors that each have capacitance C.

   Note: in your answer, please list the distinct capacitances sorted from smallest to largest values, as rational integers and as floating point numbers with three-significant digits in units of C. For each distinct value, also give the corresponding degeneracy, which is the integer number of distinct 5-capacitor arrangements that have the same effective capacitance. For example, the smallest possible 5-capacitor capacitance is $(1/5)C = 0.2C$ which has a degeneracy of 1.

5. In terms of the plate areas A, separation d (for the left capacitor) and separations d_1 and d_2 on the right, and dielectric constants K_1 and K_2 of dielectrics that fill the space between the metal capacitor plates as shown here



   give explicit mathematical formulas for the capacitances C_1 and C_2 for respectively the capacitor on the left and for the capacitor on the right.

6. You are given a battery with emf $\mathcal{E}$, a plate capacitor with capacitance C that can be connected by wires to the battery, a red dielectric plastic block with dielectric constant $\kappa_{\text{red}} > 1$ (this is a lower-case Greek kappa, not a funny-looking English k) that can be slid into the air gap between the plates of the capacitor such that the block completely fills the gap, and a blue dielectric plastic block of identical shape whose dielectric constant κ_{blue} is larger than the red block's dielectric constant, $\kappa_{\text{blue}} > \kappa_{\text{red}}$.

   Describe a sequence of actions such as connecting the capacitor to the battery, inserting or taking out one of the colored dielectric blocks, disconnecting the battery, etc such that, at the end, you end up storing the maximum possible energy out of all possible sequences. Also give your maximum energy in terms of V, C, κ_{red}, and κ_{blue}.

7. A solid metal sphere S_1 of radius R_1 and a second solid metal sphere S_2 of radius R_2 are submerged entirely in a liquid dielectric (think motor oil) with dielectric constant $\kappa > 1$ such that their centers lie a large distance from each other, so $d \gg R_1, R_2$. Sphere S_1 and sphere S_2 are then connected by thin wires to respectively the positive and negative terminals of a battery with emf $\mathcal{E}$. Obtain and give an expression for the magnitude F of the force $\mathbf{F}$ that S_1 exerts on S_2 in terms of R_1, R_2, d, κ, and $\mathcal{E}$.

   Some suggestions: first find the capacitance of the two spheres, then the charges induced on the spheres when they are connected to the battery, and finally calculate the force between the charged spheres that arises from their electrical attraction.

8. You are given a cylindrical conducting shell of length L and of radius R_1 that is concentric with and lies inside a larger cylindrical shell of radius $R_2 > R_1$, also of length L. The length $L \gg R_2$ so

end-effects can be ignored. The volume between the inner and outer shells is completely filled with a non-conducting rigid material with dielectric constant $\kappa > 1$.

(a) Show that the capacitance of this system is given by

$$C = \frac{\kappa L}{2K} \frac{1}{\ln\left(R_2/R_1\right)}, \tag{2}$$

where K is Coulomb's constant.

Hint: add a charge $+Q$ to the inner shell and add a charge $-Q$ to the outer shell. Then use Gauss's law to deduce the electric field $\mathbf{E}(r)$ between the two shells, and then calculate the potential difference between the shells via the usual line integral:

$$\Delta V = -\int_{R_1}^{R_2} \mathbf{E} \bullet d\mathbf{l}. \tag{3}$$

You will find that ΔV is proportional to the charge Q, from which you obtain $C = Q/\Delta V$.

Note: since logarithms change little over large ranges of their arguments, Eq. (2) suggests that the length L is the key parameter to vary experimentally if one wants to achieve a variety of capacitances with this geometry.

(b) Assume that the rigid dielectric material can slide in and out of the shells without friction and assume that there are charges Q and $-Q$ on the inner and outer conducting shells respectively.

i. Find an expression for the work $W(x)$ needed to pull the dielectric material out by a distance x from between the shells, with $x = 0$ corresponding to the dielectric entirely filling the region between the two conductors. From your expression, then show that the energy needed to pull the dielectric material completely out from between the cylindrical shells is given by

$$W_{\text{remove}} = \frac{KQ^2}{L}\left(1 - \frac{1}{\kappa}\right)\ln\left(\frac{R_2}{R_1}\right). \tag{4}$$

ii. Since dielectric constants are greater than 1, $\kappa > 1$, Eq. (4) is a positive expression which means that it takes energy to pull the dielectric out, which in turn means that there must be a force pulling the dielectric into the capacitor. Explain briefly where this force comes from.

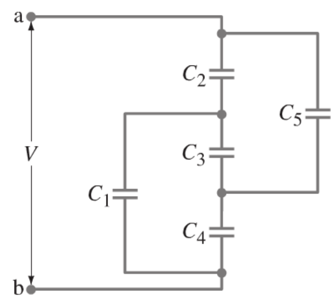
(c) What expression do you obtain instead of Eq. (4) if now a constant potential ΔV_0 is maintained across the inner and outer cylindrical shells as the dielectric is pulled out? Does the force acting on the dielectric pull the material into the capacitor or expel it?

9. **Time to Complete This Assignment**

To the nearest integer, please give the time in hours that it took you to complete this assignment.

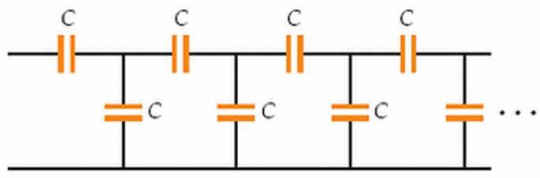
10. **Optional Challenge Problems**

(a) Challenge problem 5.1: what is the effective capacitance between points a and b on the left of the following arrangement of five capacitors



no two of which are in series or in parallel? Your answer will be some algebraic expression in terms of the five values C_1 through C_5.

(b) Challenge problem 5.2: What is the effective capacitance between the two leads on the left for this arrangement of identical capacitors that continues infinitely far to the right?



(c) Challenge problem 5.3: Consider a cube whose edges all consist of identical capacitors with capacitance C that are soldered together at each vertex of the cube. Find the effective capacitance $C_{\rm eff}$ of this cube for three cases: between a vertex and its nearest neighbor, between a vertex and its next-nearest neighbor, and between two diagonally opposite vertices.

(d) Challenge problem 5.4: explore Problem 5 further: how many distinct capacitances can be achieved with 6 identical capacitors? How many of these configurations can not be reduced to series and parallel combinations of capacitances? More generally, how does the number of distinct capacitances grow with the number N of identical capacitors used in a given circuit? Given a box of arbitrarily many identical capacitors with $C = 1\,\mathrm{F}$, can you create an effective capacitance arbitrarily close to any positive real number? (In mathematical language, are the set of values of capacitances that you can produce with finitely many identical capacitances dense in the positive real line?)

Note: some parts of this problem are not only hard but unsolved, for example, a formula is not known for the number $n_C(N)$ of distinct capacitances that can be created with N identical capacitances. So consider this a good exploratory problem and write up and give me any progress you make on this problem.