Physics 162: Assignment 4

Made available: Monday, February 2, 2015
Due: Sunday, February 8 by 10 am.

1. **Tutorial on the gradient ∇f of a multivariable function $f(x, y, z)$**

This problem is a tutorial that gives you some practice with a few concepts from multivariable calculus. Galileo Galilei said that "Mathematics is the language with which God has written the universe." and the mathematical concepts of gradient, curl, and divergence are key parts of the language that describes electromagnetism (and fluids, and plasmas, and quantum mechanics, Galileo was very much right).

Most of this problem involves reading and then carrying out a few modest calculations that are indicated in bold print below.

Although this first problem may seem long and possibly intimidating, it is actually quick and easy since all you need to know to answer the problems is how to evaluate the partial derivative $\partial f/\partial x$ of some function, which is also often denoted as $\partial_x f(x, y, z)$. This in turn is the same as taking an ordinary derivative d/dx of some expression by treating all symbols in the expression as constants except the differentiation symbol x.

We will not be using the mathematics of this tutorial substantially this semester but it is good for you to start to become familiar with these mathematical ideas. Especially important is the insight that the electric field $\mathbf{E}(x, y, z)$ of a static arrangement of charges can be expressed as the gradient $-\nabla V$ (some kind of derivative) of a scalar field $V(x, y, z)$ called the electric potential, and it is often more productive and insightful to think about electric field problems first in terms of V, and first to carry out calculations in terms of V since it is easier to work with a scalar field than with a vector field. You should also appreciate that there is a precise easy-to-compute mathematical criterion to determine whether some vector field is conservative, see Eq. (19) below.

So let's begin the tutorial. An important mathematical concept is the *gradient* of some scalar field $f(x, y, z)$, denoted as the vector field $\nabla f(x, y, z)$ and pronounced in English as "grad f". (The symbol ∇ is called "nabla", and sometimes people write $\mathbf{grad}(f)$ instead of ∇f.) You can think of the gradient as a way to convert a function f that assigns a number to any given point in space (x, y, z) into a vector function $\nabla f(x, y, z)$ that assigns a vector to any point in space. The gradient generalizes the derivative df/dx of a scalar function $f(x)$ of a single variable to the derivative of a scalar function $f(x, y, z)$ of several variables.

The gradient is important for electromagnetism in that it connects[1] a concept related to energy, the electric potential $V(x, y, z)$, to the electric field $\mathbf{E}$ via the relation $\mathbf{E} = -\nabla V$. The gradient is also important in that it occurs in the second term of the multivariable Taylor series expansion of some scalar function $f(x, y, z) = f(\mathbf{x})$ about some point $\mathbf{x}_0$ of interest

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \bullet (\mathbf{x} - \mathbf{x}_0) + \cdots , \tag{1}$$

which you should compare with the one-dimensional version

$$f(x) = f(x_0) + \frac{df}{dx}(x_0)(x - x_0) + \cdots . \tag{2}$$

Because multivariate Taylor-series approximations like Eq. (1) are so widely used in many areas of physics, engineering, and mathematics (to approximate some complicated function locally with a simpler linear function) gradients are a very important tool in mathematics. Also, just as $df/dx = 0$

[1] Something may seem funny here, that a single function $V(x, y, z)$ can generate three functions $E_x(x, y, z)$, $E_y(x, y, z)$ and $E_z(x, y, z)$ corresponding to the components of the electric field via $\mathbf{E} = -\nabla V$. But the components of a *static* electric field are not independent because static electric fields are conservative which, according to Eqs. (18) and (19) below, implies the three conditions $\partial_y E_z = \partial_z E_y$, $\partial_z E_x = \partial_x E_z$, and $\partial_x E_y = \partial_y E_x$. So static electric fields can not be arbitrary vector fields since the components are related to one another although in a rather non-obvious way.

indicates where a function of one variable has an extremum (local minimum, local maximum, or inflection point), Eq. (1) implies that the extrema of a multivariable scalar function are determined by the algebraic conditions $\nabla f = \mathbf{0}$, which is of great importance in many fields when trying to optimize some scientific, engineering, or economic problem.

Given a differentiable scalar function $f(x, y, z)$, its gradient ∇f is defined to be the vector field

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \tag{3}$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right). \tag{4}$$

For two scalar functions $f(x, y, z)$ and $g(x, y, z)$ and for constants c_1 and c_2, show that

$$\nabla(fg) = f\nabla g + g\nabla f \quad \text{and} \quad \nabla\left(c_1 f + c_2 g\right) = c_1 \nabla f + c_2 \nabla g, \tag{5}$$

Eq. (5) says that the gradient operator ∇ has properties similar to those of the one-dimensional derivative d/dx applied to a product of functions $f(x)g(x)$ and to a linear combination of functions $c_1 f + c_2 g$.

From a previous physics or math course, you know that the product of a number c times a vector $\mathbf{a}$ is defined in terms of the vector components like this:

$$c\,\mathbf{a} = c\,(a_x, a_y, a_z) = (ca_x, ca_y, ca_z)\,. \tag{6}$$

Since the usual multiplication of numbers doesn't depend on the order, $xy = yx$ for any two real numbers, you might also be motivated to explore defining the product of a vector times a number. I hope you would agree that the following definition is reasonable

$$\mathbf{a}\,c = (a_x, a_y, a_z)\;c = (ca_x, ca_y, ca_z)\,, \tag{7}$$

since a vector times a number should be a vector and the right side of Eq. (7) is a rather obvious choice.

Given Eqs. (6) and (7), you might be motivated to explore writing the definition Eq. (3) also as a vector times a number, like this:

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} = \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right) f, \tag{8}$$

which indeed looks like a vector (a rather funny looking vector since it has derivatives) times some number-valued function f. From this, you might be encouraged to guess that the expression

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}, \tag{9}$$

could be treated as a vector in its own right. This indeed turns out to be a fruitful idea. This kind of expression involving derivatives is called an "operator" since it operates on some function f to give some new expression involving functions[2].

If ∇ as defined by Eq. (9) is a vector, you might now wonder what is the dot product of this vector with some vector function such as

$$\mathbf{f} = f_x(x, y, z)\,\hat{\mathbf{x}} + f_y(x, y, z)\,\hat{\mathbf{y}} + f_z(x, y, z)\,\hat{\mathbf{z}}. \tag{10}$$

[2]If you continue on with physics or biophysics to take quantum mechanics, you will learn that, in the quantum theory, experimental observables are represented by operators of this kind which is strange but very cool. For example, momentum in quantum mechanics is represented by the vector operator $\mathbf{p} = i\hbar\nabla$.

Since derivatives act on functions, there seems to be little choice but to put ∇ on the left side of $\mathbf{f}$ and so we might guess that the expression

$$\nabla \bullet \mathbf{f} = \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}} \right) \bullet \left(f_x\hat{\mathbf{x}} + f_y\hat{\mathbf{y}} + f_z\hat{\mathbf{z}} \right) \tag{11}$$

$$= \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}, \tag{12}$$

is the way to define the dot product of nabla with a vector function.

Note that the dot product of two vectors must be a number and indeed the right side of Eq. (12) is a number as a sum of number-valued functions. Note also that, while the order of the dot product of two ordinary vectors $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$ doesn't matter, the order $\mathbf{f} \bullet \nabla$ does not mean the same thing as $\nabla \bullet \mathbf{f}$ since $\mathbf{f} \bullet \nabla = f_x\partial_x + f_y\partial_y + f_z\partial_z$ is an operator containing several derivatives that do not have meaning until they operate on some some specific function. For example, $(\mathbf{f} \bullet \nabla)g(x, y, z) = f_x\partial_x g + f_y\partial_y g + f_z\partial_z g$.

The right side of Eq. (12) seems reasonable and so people *define* the dot product $\nabla \bullet \mathbf{f}$ (pronounced as "div ef") to be

$$\nabla \bullet \mathbf{f} = \mathrm{div}(\mathbf{f}) = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}. \tag{13}$$

This dot product is also called the *divergence* of the vector function $\mathbf{f}$ and is often written as div($\mathbf{f}$), which is why $\nabla \bullet f$ is pronounced "div ef". Eq. (13) has the physical meaning of the total flux per unit volume of the vector field ∇f into an infinitesimal volume centered on the point (x, y, z).

Show that for some scalar function $f(x, y, z)$, the divergence of the gradient of f (pronounced "div grad f") is given by the following expression:

$$\nabla \bullet (\nabla f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{14}$$

Since the dot product of a vector with itself $\mathbf{a} \bullet \mathbf{a}$ is often written as a^2 (magnitude a of the vector squared), the scalar combination of second-order partial derivatives Eq. (14) is often written as $\nabla^2 = \nabla \bullet \nabla$ in which case $\nabla^2 f$ is called "the Laplacian of the scalar function $f(x, y, z)$" The Laplacian operator is quite important and shows up in many areas of physics, engineering, applied mathematics, and computer science[3].

Compute the Laplacian of a general quadratic polynomial in x, y, and z:

$$\nabla^2 \left(c_1 + c_2 x + c_3 y + c_4 z + c_5 xy + c_6 xz + c_7 yz + c_8 x^2 + c_9 y^2 + c_{10} z^2 \right) = ? \tag{15}$$

where c_1 through c_{10} are constants.

OK, we are on a roll. After defining the dot product of the gradient operator ∇ with some vector function, you may next be curious how to define the cross product of the vector ∇ with some vector function $\mathbf{f} = (f_x, f_y, f_z)$. This should give a vector-valued expression since the cross product $\mathbf{a} \times \mathbf{b}$ of two vectors is a vector. Let's continue to run with the idea of thinking of the gradient as a vector given by Eq. (9). Remembering that the cross product of two vectors $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$ is defined to be

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\,\hat{\mathbf{x}} + (a_z b_x - a_x b_z)\,\hat{\mathbf{y}} + (a_x b_y - a_y b_x)\,\hat{\mathbf{z}}, \tag{16}$$

and thinking of $\mathbf{a}$ as the vector ∇, we could try making the substitutions

$$a_x \to \frac{\partial}{\partial x}, \quad a_y \to \frac{\partial}{\partial y}, \quad a_z \to \frac{\partial}{\partial z}, \tag{17}$$

[3]For example, the electric potential $V(x, y, z)$ associated with some system with charge density $\rho(x, y, z)$ satisfies a partial differential equation equation called Poisson's equation, which looks like this: $\nabla^2 V = -\rho/\epsilon_0$. Poisson's equation is actually the best practical way to calculate the electric field $\mathbf{E}$ for complicated charges by first calculating V from Poisson's equation (which can be solved rapidly and accurately with modern computer codes) and then using the fact that $\mathbf{E} = -\nabla V$.

to get the following tentative definition of the cross product of ∇ with some vector function $\mathbf{f}$:

$$\nabla \times \mathbf{f} = \mathbf{curl}(\mathbf{f}) = \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right) \hat{\mathbf{z}}. \tag{18}$$

This cross product is pronounced in English as either "grad cross f" or more often[4] as "the curl of f" since many books write $\nabla \times \mathbf{f}$ as the vector field $\mathbf{curl}(\mathbf{f})$.

Although Eq. (18) appears complicated, you really only have to memorize the first component $\partial_y f_z - \partial_z f_y$ since the other components are obtained by a cyclic substitution of indexes: $x \to y \to z \to x$.

Now that you know about the curl of a function, I can tell you without proof that a vector function $\mathbf{f}(x, y, z)$ is conservative in some region of space if and only if its curl vanishes in that region of space:

$$\mathbf{f} \text{ is conservative} \quad \Longleftrightarrow \quad \nabla \times \mathbf{f} = 0. \tag{19}$$

As a technical aside, I mention briefly that Eq. (19) is a consequence of an important theorem of multivariable calculus known as "Stoke's theorem" which relates a line integral $\oint_\Gamma$ over some closed curve Γ to a surface integral over any surface S spanning the curve Γ:

$$\oint_\Gamma \mathbf{f} \bullet d\mathbf{l} = \int_S (\nabla \times \mathbf{f}) \bullet d\mathbf{A}. \tag{20}$$

(If you want to learn more, one of the clearest and most insightful discussions of Stoke's theorem that I can recommend is Chapter 2 of the book "Electricity and Magnetism, third edition" by E. Purcell and D. Morin.) If the vector field $\nabla \times \mathbf{f}$ vanishes in some region of space, the line integral over any closed loop is zero in that region of space. This then implies that the line integral between two points is independent of the path connecting those points, since any two paths connecting the same two points constitute a closed loop to which you can apply Stoke's theorem.

(a) **Show that any vector field $\mathbf{f} = \nabla g$ that can be written as the gradient of some scalar field $g(x, y, z)$ is conservative:**

$$\nabla \times (\nabla g) = 0. \tag{21}$$

(b) **Show that the vector function**

$$\mathbf{f} = \left[2xy - z^3\right] \hat{\mathbf{x}} + \left[x^2\right] \hat{\mathbf{y}} + \left[-\left(3xz^2 + 1\right)\right] \hat{\mathbf{z}}, \tag{22}$$

is a conservative force field, that is $\nabla \times \mathbf{f} = 0$.

Eq. (21) implies the useful result that any central vector field of the form $g(r)\,\hat{\mathbf{r}}$, that is such that the magnitude depends only on the distance $r = \sqrt{x^2 + y^2 + z^2}$ from the origin and that points along radial lines $\hat{\mathbf{r}} = (x, y, z)/\sqrt{x^2 + y^2 + z^2}$ must be conservative. To see this, define a new function $f(r)$ in terms of the given function g by

$$f(r) = \int_0^r g(r')\, dr'. \tag{23}$$

You should then be able to show that $g(r)\hat{\mathbf{r}} = \nabla f(r)$ has a gradient form. Coulomb's law and Newton's gravity law, which give central forces, then implies that static electric fields and gravitational fields are conservative.

There is a converse result of importance for understanding electrostatic fields: if $\nabla \times \mathbf{f} = 0$ over some region of space, there exists a function g in that region such that $\mathbf{f} = \nabla g$. The function $g(x, y, z)$ can be determined by evaluating a line integral of $\mathbf{f}$ from some fixed arbitrary reference point $\mathbf{p}_0 = (x_0, y_0, z_0)$ to the point (x, y, z) using any continuous path. So if

$$g(x, y, z) = g(\mathbf{x}) = \int_{\mathbf{p}_0}^{\mathbf{x}} \mathbf{f} \bullet d\mathbf{l}, \tag{24}$$

[4] The "curl" name comes from taking the curl of a fluid velocity field $\mathbf{v}(t, x, y, z)$, in which case the physical meaning of Eq. (18) is how much the fluid is swirling or curling around some point of interest in the fluid.

then I claim that $\mathbf{f} = \nabla g$.

For f defined by Eq. (22), use Eq. (24) to determine the "potential" g such that $\mathbf{f} = \nabla g$, then verify your result by showing directly that $\nabla g = \mathbf{f}$.

A hint is to make the evaluation of the line integral on the right side of Eq. (24) easy by choosing the reference point $\mathbf{p}_0 = (0,0,0)$ to be the origin, and then by using the fact that $\mathbf{f}$ is conservative so you can use any path that connects the origin to a general point (x,y,z). One convenient path is one that integrates along successive line segments that are always parallel to the coordinate axes, this involves only one component of $\mathbf{f}$ at a time and integrates over only one variable at a time. For example, first use the straight path that goes from $(0,0,0)$ to $(x,0,0)$ along the x-axis. Along this path, the differential vector $d\mathbf{l}$ becomes $d\mathbf{l} = dx\,\hat{\mathbf{x}}$ which points only along the x-axis, $\mathbf{f} \bullet d\mathbf{l} = f_x = 2xy - z^3$ becomes only the x-component of the force field $\mathbf{f}$, and the line integral in Eq. (24) along just this first part of the path becomes:

$$\int_{(0,0,0)}^{(x,0,0)} \mathbf{f} \bullet d\mathbf{l} = \int_{(0,0,0)}^{(x,0,0)} \mathbf{f} \bullet (dx'\,\hat{\mathbf{x}}) = \int_{(0,0,0)}^{(x,0,0)} \left(2x'y - z^3\right)\, dx'. \tag{25}$$

This integral actually vanishes since $y = 0$ and $z = 0$ along the x-axis and so the integrand itself vanishes along this path. Next integrate from $(x,0,0)$ to $(x,y,0)$ along the line parallel to the y axis, in which case the length differential becomes $d\mathbf{l} = dy\,\hat{\mathbf{y}}$ and so

$$\int_{(x,0,0)}^{(x,y,0)} \mathbf{f} \bullet d\mathbf{l} = \int_{(x,0,0)}^{(x,y,0)} \mathbf{f} \bullet (dy'\,\hat{\mathbf{y}}) = \int_{(x,0,0)}^{(x,y,0)} f_y\, dy' = \int_{(x,0,0)}^{(x,y,0)} x^2\, dy', \tag{26}$$

and you hold x constant when you integrate w.r.t. y'. The last part of the line integral will be along a vertical line segment parallel to the z axis, so $d\mathbf{l} = dz\,\hat{\mathbf{z}}$ and $\mathbf{f} \bullet d\mathbf{l} = f_z dz$ along this line, giving

$$\int_{(x,y,0)}^{(x,y,z)} \mathbf{f} \bullet d\mathbf{l} = \int_{(x,y,0)}^{(x,y,z)} \mathbf{f} \bullet (dz'\,\hat{\mathbf{z}}) = \int_{(x,y,0)}^{(x,y,z)} f_z\, dz' = -\int_{(x,y,0)}^{(x,y,z)} 3x(z')^2\, dz'. \tag{27}$$

Your final answer will be the sum of the three values along each path:

$$\int_{(0,0,0)}^{(x,y,z)} \mathbf{f} \bullet d\mathbf{l} = \int_{(0,0,0)}^{(x,0,0)} \mathbf{f} \bullet d\mathbf{l} + \int_{(x,0,0)}^{(x,y,0)} \mathbf{f} \bullet d\mathbf{l} + \int_{(x,y,0)}^{(x,y,z)} \mathbf{f} \bullet d\mathbf{l}. \tag{28}$$

2. **Deducing E from V** Three identical point charges, each with a charge equal to q, lie in an xy-coordinate plane. Two of the charges are on the y-axis with coordinates $y = \pm a$, and the third charge is on the x-axis with location $x = a$.

   (a) Find the potential $V(x)$ as a function of position along the x axis, and plot it qualitatively.

   (b) Use your result for V to calculate the x-component of the electric field via $E_x = -dV/dx$, and verify the correctness of your result by calculating E_x directly for this problem via superposition. Note: you will need to write two different expressions for the cases $x > a$ and $x < a$.

3. **What does it really mean to ground a conductor?** Consider two concentric spherical conducting surfaces of radius R_1 and radius $R_2 > R_1$ such that the inner surface has a total charge Q and the outer surface has an opposite charge of $-Q$.

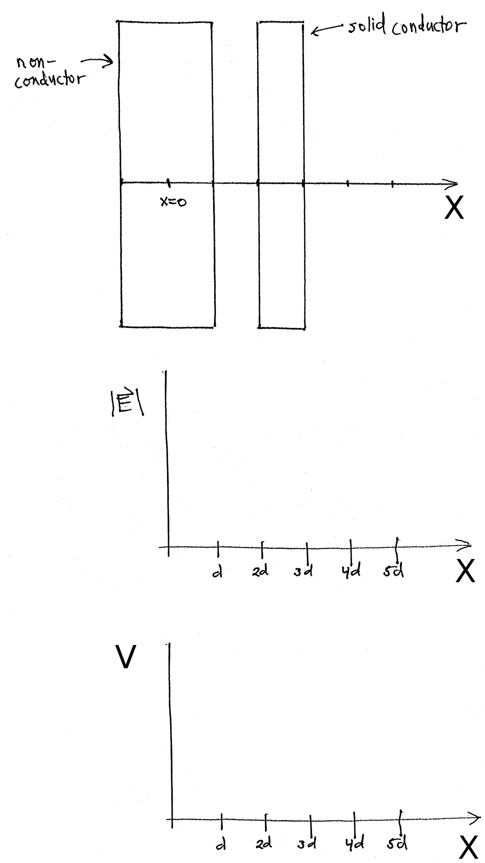
   (a) If a ground wire is connected to the outer surface, explain why no charge will move off or onto the outer surface.

(So grounding a conductor does *not* always eliminate its charge. The situation would also be different if the two spheres were placed side, rather than one inside the other, then grounding one of the spheres will cause some charge to transfer.)

(b) If instead a ground wire is connected to the inner surface (say by passing a thin ground wire through a tiny hole made in the outer surface), determine the amount of charge Q_i (in terms of Q, R_1, and R_2) on the inner surface after electrostatic equilibrium is attained.

Some hints: grounding a conductor, say by connecting it to a wire that is itself electrically connected to the Earth's soil, forces the electric potential V of the conductor to have the same value $V = 0$ as a point arbitrarily far away from the conductor since the Earth basically has an electric potential of $V = 0$. This implies that, when a ground wire is touched to a conductor, charge will move onto or off the grounded conductor until its potential difference $\Delta V = V(\text{conductor}) - V(\infty) = V(\text{conductor}) = 0$. So to solve this problem, you need to evaluate some line integrals between infinity and the grounded conductor, and adjust the charge on the grounded conductor until its electric potential is zero.

4. **Potential of charged plane and parallel conducting plate** A long planar slab of thickness $2d$ consists of a non-conducting material with constant negative charge density $\rho < 0$. At a distance d to the right of the non-conducting slab, a neutral long planar solid conductor of thickness d is placed as shown in the figure on the following page, which shows the slabs edge on:



You can assume that the slabs are so long compared to their width that you can ignore fringing effects of the fields at their ends.

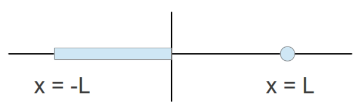
(a) On a similar bigger diagram in your homework, show the direction of the $\mathbf{E}$ field at locations $x = 0$, $0.5d$, $1.5d$, $2.5d$, and $4d$. Then calculate and give mathematical expressions for the magnitude $|E|$ for $0 \le x \le 5d$, and plot your magnitude on the given axes.

(b) Calculate and give mathematical expressions for the electric potential $V(x)$ for $0 \le x \le 5d$ with the reference point chosen to be $V(x = 0) = 0$. Then plot your $V(x)$ expression on the given axes, and discuss briefly how or why your plot makes physical sense.

5. **An unstable system of two positrons and two protons** Two positrons (positively charged antiparticles of the electron, which have the same mass as an electron) lie at opposite corners of a square of side $L = 1\,\mathrm{cm}$. The other two corners of the square are occupied by protons. Initially the four particles are held in these positions at rest and then are released simultaneously. When the particles are all far away from each other, explain why the speeds of the positrons will be about $350\,\mathrm{m/s}$ while the speeds of the protons will be about $3\,\mathrm{m/s}$.

   Hint: a positron has a mass that is about 1/2000 the mass of a proton so think about the relative accelerations of these particles right after they are released.

6. **Potential due to a charged rod and point charge** A thin rod of length L is placed on the x-axis of an xyz-Cartesian coordinate system so that its ends lie at coordinates $x = -L$ and $x = 0$, and then an amount Q of positive charge is spread uniformly over the rod. A point particle with positive charge Q is then placed at coordinate $x = L$ as shown in this figure:



(a) Show that the location on the x-axis between the rod and point charge where the electric field vanishes is $x = L/3$. (Note that this is *not* the answer that you would get if you replaced the rod by a point particle of charge Q at its center.) Are there any other places in space where $\mathbf{E} = \mathbf{0}$?

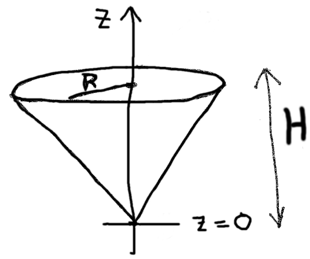
(b) Assuming $V(\infty) = 0$ so that it is ok to use the formula Kq/r for the electric potential of a point that is a distance r from a point charge q, calculate and given an expression for the potential $V(x)$ of this "rod plus charge" system for $x < -L$ and $x > 0$ and give a qualitative plot of $V(x)$ over the range $[-2L, 2L]$.

   Discuss briefly whether it is meaningful to talk about the potential $V(x)$ inside the rod itself.

(c) A point particle with positive charge q and mass m is placed at the location $x = L/3$ where $\mathbf{E} = \mathbf{0}$ and given a tiny push. Show that a sufficiently long time later, the speed v of the particle is given by

$$v = \left[\frac{2KqQ}{mL} \left(\frac{3}{2} + \ln 4 \right) \right]^{1/2} . \tag{29}$$

7. **Potential difference across a charged ice cream cone** A non-conducting surface in the shape of a hollow ice-cream cone (no top and no ice cream) with height H and radius R has a constant surface charge density σ:



(a) Obtain and give a mathematical expression in terms of H, R, σ, and z for the potential difference $V(z) - V(0)$ between a point on the axis of the cone with coordinate z and the tip of the cone for which $z = 0$. You should evaluate any integrals in your expression (it is ok to use Mathematica or Wolfram Alpha for this, although this is also a valuable chance to practice your integration skills). Can you tell from your answer for what z you get the largest potential difference $|V(z) - V(0)|$?

Also discuss briefly some limits of your expression that help to convince you that your final answer is scientifically reasonable.

(b) For a cone that you could make by cutting a sector of a circle out a sheet of notebook paper, rolling the sector up into a cone, and then charging the cone with a rubber rod that has been rubbed with cat fur, estimate to the nearest power of ten the largest potential difference $|V(z) - V(0)|$ you would expect between a point on the axis and the tip of the cone. Make sure to explain any assumptions or approximations you make in getting your answer.

Note: do not use any electric fields in this problem, instead start with the known formula of Example 28.12 on page 829 of Knight for the potential of a point (with respect to infinity) that lies on the axis of a uniformly charged ring of known radius. You may also want to use a result known to the Greeks, that if you rotate a line segment of length l around a radius of length R that passes through the center of the line segment, the line segment sweeps out an area $S = (2\pi R)l$. This formula is a straightforward deduction from the formula for the surface area $S = \pi RL$ of a cone of radius R at its base and of side length L, see the video tutorial `https://www.youtube.com/watch?v=K2ghejiUDXg` for the pretty insight of how to get this formula.

8. **Time to Complete This Assignment**

To the nearest integer, please give the time in hours that it took you to complete this assignment.