

Physics 162: Assignment 3

Made available: Tuesday, January 27, 2015

Due: Saturday, January 31 by 7 PM.

Problems

1. By direct evaluation of an appropriate one-variable integral, show that the electric flux $\Phi = \oint \mathbf{E} \cdot d\mathbf{A}$ through a circle of radius R due to a point charge Q lying on the axis through the center of the circle, a distance d from the center of the circle, is given by

$$\Phi = \frac{Q}{2\epsilon_0} \left(1 - \frac{d}{\sqrt{d^2 + R^2}} \right) \quad (1)$$

Explain why this formula is reasonable by giving two limits for which you know the answer by other means.

Note: you will be using this result later in the course, when we discuss ways to detect a magnetic monopole experimentally. But the real purpose of this problem is to show you how much effort is required to calculate the flux of an electric field through some surface when the electric field is not everywhere parallel or perpendicular to that surface.

Some suggestions: assume that the circle is centered on the origin of an xyz -coordinate system and lies in the xy -plane, and further assume that the point charge Q has location $(x, y, z) = (0, 0, d)$ in this coordinate system. Now decompose the circle into concentric infinitesimal rings labeled by a radius r , each of equal infinitesimal thickness dr , calculate the flux $d\Phi$ through each such ring, and add up all the little fluxes by integration. The key point you need to take into account is that the angle of the electric field with the normal to each infinitesimal ring varies with r .

2. An infinitely long nonconducting cylindrical shell of inner radius r_1 and outer radius $r_2 > r_1$ has a uniform volume charge density ρ .
 - (a) Explain why the electric field of this object is everywhere radial, pointing away or towards the axis of the shell from some given point of interest. ("Everywhere" includes inside the material of the shell.)
 - (b) Use Gauss's law to show that the magnitude $E(r)$ of the electric field at a distance $r \geq 0$ from the axis of shell is given by:

$$E(r) = \begin{cases} 0, & r < r_1 \\ \frac{\rho}{2\epsilon_0} \left(r - \frac{r_1^2}{r} \right), & r_1 \leq r < r_2 \\ \frac{\rho (r_2^2 - r_1^2)}{2\epsilon_0} \frac{1}{r}, & r_2 \leq r. \end{cases} \quad (2)$$

Explain why this expression is reasonable by taking various limits for which you know the answer by other means.

- (c) Finally, show that you understand the meaning of Eq. (2) by drawing a qualitative graph of $E(r)$ versus r over the range $0 \leq r \leq 2r_2$ for the case of a positive charge density $\rho > 0$ and for $r_2 = 2r_1$. Make sure to indicate the locations $r = r_1$ and $r = r_2$ clearly on your graph.
3. A uniformly charged non-conducting solid sphere of radius R and charge density ρ has its center at the origin of an xyz coordinate system

- (a) Use Gauss's law to show that at a point P *inside* the sphere that is a distance r from the center, the electric field is given by

$$\mathbf{E} = \frac{\rho}{3\epsilon_0} r \hat{\mathbf{r}}, \quad (3)$$

where the unit vector $\hat{\mathbf{r}}$ points from the origin to P .

- (b) A spherical cavity of radius R' is now scooped out from inside the sphere at an arbitrary location. Show that the electric field inside the cavity is everywhere uniform with value

$$\mathbf{E}_{\text{cavity}} = \frac{\rho}{3\epsilon_0} \mathbf{a}, \quad (4)$$

where the vector \mathbf{a} is the vector pointing from the center of the solid sphere to the center of the spherical cavity.

Hint: use superposition to reduce the sphere with cavity to two problems that you know how to solve more easily.

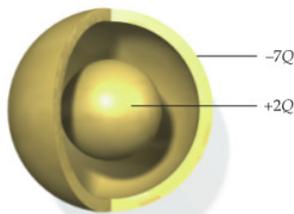
4. Assume that you are a scientist living before the 1911 experiment of Geiger and Marsden, in which the scattering of alpha particles (emitted by a radioactive substance) off a thin gold sheet demonstrated the remarkable result that gold atoms consisted of a tiny massive positive nucleus surrounded somehow by negatively charged electrons. So you know about positive and negative charges but you don't know how they are arranged inside of an atom or molecule, and you don't know yet that electrons and nuclei are point-like particles.

In this problem, you use Gauss's law several times to develop a crude but insightful classical (non-quantum) "jelly" theory of a helium atom (also of a diatomic hydrogen molecule H_2) by assuming that the helium atom consists of a negatively-charged spatially-uniform rigid spherical jelly of radius R and of total charge $-2e$, combined somehow with two point positive charges (protons). Since helium atoms are stable objects (they don't spontaneously fall apart, even if bumped), you guess that a helium atom may correspond to some stable mechanical equilibrium of the jelly and the two protons, such that the total force on each proton is zero.

- (a) Explain why no mechanical equilibrium can occur if both protons lie outside the negative electron jelly.
- (b) Explain why no equilibrium can occur if one proton is inside the negative jelly and one proton is outside.
- (c) You therefore conclude that both protons must be inside the electron jelly for a mechanical equilibrium to occur. Explain why there is only one possible equilibrium of two protons that are both inside the jelly, which is when the two protons lie along the same diameter, on either side of the center, and equal distances from the center.
- (d) Finally, show that in the unique equilibrium state, the distance d of the protons from the center of the electron sphere of radius R is given by $d = R/2$.

Note: remember our discussion earlier this semester that it is impossible to have a stable static equilibrium of points charges, which therefore cannot provide a way to explain the structure of atoms or molecules. (This result is known as "Earnshaw's theorem" if you want to look it up.) It turns out (see the challenge problem 3.3 below) that Earnshaw's theorem does not apply to spatially extended jelly-like charged objects for which it is possible to have stable mechanical equilibria, and so this provides a theoretical loop-hole by which one could develop a theory of a molecule, provided that either the positive or negative charge is spatially extended. But the Geiger and Marsden experiment showed that spatially extended charges did not exist in actual atoms and so something new and drastic was needed to understand atoms, and this turned out to be quantum mechanics.

5. Consider a solid conducting sphere and concentric conducting shell as shown in this figure:



The spherical shell has a total negative charge of $-7Q$ while the solid sphere has a total positive charge $2Q$, and the system is in electrostatic equilibrium.

- (a) Use Gauss's law to explain why the charge on the shell's outer surface is $-5Q$ and why the charge on the shell's inner surface is $-2Q$.
- (b) Suppose a metal wire is now connected between the solid sphere and the shell. After electrostatic equilibrium is reestablished, explain why the charge on the solid sphere is now zero, the charge on the inner surface of the shell is 0, and the charge on the outer surface is still $-5Q$. Discuss also how the electric field between the sphere and shell has changed.
- (c) The system is now returned to its original state with a charge of $2Q$ on the sphere and a charge of $-7Q$ on the shell. After a grounded wire is briefly touched to the shell and removed, what is the charge on the sphere, the charge on the shell's inner surface, and charge on the shell's outer surface?
6. An infinite slab of a non-conducting material of thickness $2d$ fills the region $-d \leq y \leq d$ of an xyz -coordinate system. The material in the region $-d \leq y < 0$ has uniform charge density $\rho > 0$, while the material in the region $0 \leq y < d$ has uniform charge density $-2\rho < 0$.

- (a) Explain briefly why Gauss's law cannot be applied directly to this charged object.
- (b) By an appropriate use of superposition, use Gauss's law to show that the electric field vector $\mathbf{E}(y)$ is given everywhere in space by the following expression:

$$\mathbf{E}(y) = \begin{cases} \frac{\rho}{2\epsilon_0} d \hat{\mathbf{y}}, & y \leq -d \\ \frac{\rho}{2\epsilon_0} (3d + 2y) \hat{\mathbf{y}}, & -d \leq y < 0 \\ \frac{\rho}{2\epsilon_0} (3d - 4y) \hat{\mathbf{y}}, & 0 \leq y < d \\ -\frac{\rho}{2\epsilon_0} d \hat{\mathbf{y}}, & d \leq y \end{cases} \quad (5)$$

- (c) Show that you understand conceptually what Eq. (5) means by drawing a sketch of the electric field component along the y -axis, $E_y(y)$ versus y (*not* the magnitude!), over the range $-2 \leq y \leq 2$ for the parameter values $d = 1$ and $\rho = 2\epsilon_0$.
7. A solid conducting slab of dimensions $1 \text{ cm} \times 5 \text{ m} \times 5 \text{ m}$ contains a net charge of $Q = 80 \mu\text{C}$. Nearby and parallel to the plane of the slab lies an infinite non-conducting plane with surface charge density $\sigma = 2.0 \mu\text{C}/\text{m}^2$.
- (a) Find the amount of charge on each side of the square slab and find the directions and magnitudes of the electric field at points that are just outside the surface of the slab, over its center. A hint is that the total electric field inside any conductor that is in electrostatic equilibrium must be zero. You should also find that the charge on one side of the slab is $15 \mu\text{C}$ and that

the magnitude of one of the electric field vectors just outside the center of the slab is about $7 \times 10^4 \text{ N/C}$.

- (b) Show that the total force acting on the slab is about 9 N and determine the direction of the force. Dividing the force by area gives an average pressure acting on the slab, give this pressure in units of atmospheric pressure at sea level to the nearest power of ten. Is the pressure large or small compared to atmospheric pressure.

8. Time to Complete This Assignment

To the nearest integer, please give the time in hours that it took you to complete this assignment.

9. Optional Challenge Problems

- (a) Challenge 3.1: How should one define the electric flux through a Möbius strip?
- (b) Challenge 3.2: Consider a long neutral cylinder-like conducting shell whose cross-section is given in this figure (white areas denote vacuum, shaded areas denote metal):



A long line of uniform positive charge is now placed along the axis of this shell (this appears in cross section as the black dot in the figure). Describe the resulting charge distribution of this shell: which of the six surfaces have negative surface charge densities, which have positive surface charge densities, and which are neutral?

- (c) Challenge 3.3: For the jelly-like model of a He atom or of a diatomic hydrogen molecule in Problem 4 above:
- Show that your equilibrium state is stable, which means that a tiny displacement of either proton from its equilibrium position in *any* direction leads to a restoring force that points back towards the proton's equilibrium position.
 - Calculate the frequency f of oscillation of a proton about its stable equilibrium position and compare your value to the vibrational frequency of a proton in a neutral diatomic hydrogen molecule H_2 . Does the jelly model give even the right order-of-magnitude value for the frequency?
 - Try developing a jelly-theory of the periodic table by extending Problem 4 to the case of other atoms with more protons. Can you find a stable mechanical equilibrium of three point protons in a uniform spherical jelly of charge $-3e$, corresponding to a lithium atom? How about for beryllium (four protons)? Can you find an atom for which multiple distinct stable equilibria exist? Do all neutral atoms have stable equilibria for arbitrarily many point protons?