

Physics 162: Assignment 2

Made available: Saturday, January 17, 2013

Due: Friday, 11 pm, January 23, 2014

1. **Frequency of oscillation for a charged particle about a stable equilibrium** In class, we discussed the idea of a stable mechanical equilibrium of some point particle (also for some rigid object), which is a location such that the total force on the particle is zero (so the particle remains at rest if it is already at rest), and such that a small displacement of the particle in any allowed direction causes a force to appear that pushes the particle back towards its equilibrium position. A marble at the bottom of a hemispherical fruit bowl would be a simple familiar example of a stable mechanical equilibrium.

If the displacement from equilibrium is sufficiently small, one can show (by a Taylor-series expansion of the force about the equilibrium location) that the particle will undergo simple harmonic periodic motion with some characteristic period T (or equivalently with some frequency $f = 1/T$ or with some angular frequency $\omega = 2\pi f = 2\pi/T$). There are many physics and engineering problems for which it is useful or important to know or to predict what is the frequency ω of oscillation about a given equilibrium point. In this problem, you use your knowledge of Coulomb's law and electric fields to calculate the oscillation frequency for a crude model of atoms oscillating back in forth in some crystal lattice.

Two point particles with identical positive charges $q > 0$ are fixed at the coordinates $(x = 0, y = \pm L)$ of an xy -coordinate system. A long thin thread is placed along the x axis and a small bead with *negative* charge $Q < 0$ and mass m is placed on the thread so that the bead can slide along the x -axis without friction.

- (a) Without doing any mathematical calculation, think qualitatively about how the x -component of the force $F_x = QE_x$ acting on the bead must behave as a function of x , and then draw a qualitative plot of $F_x(x)$ vs x to summarize your thinking. Make sure to explain your reasoning of why your curve has particular features.
- (b) Repeat the previous problem (plot qualitatively F_x vs x) but for the case that the bead has a positive charge $Q > 0$. Explain why your plot for the case $Q < 0$ implies periodic motion near $x = 0$, while your plot for the case $Q > 0$ implies periodic motion is not possible about any value of x .
- (c) At what locations x_{\max} along the x axis does the electric field vector \mathbf{E} due to the two positive charges have the largest magnitude, and what is the value E_{\max} of that largest magnitude? As well as giving mathematical expressions for x_{\max} and E_{\max} , give the decimal values of the numerical coefficients to one significant digit so you have some intuition about the values of the numerical coefficients. (As one check, $E_{\max} = (4/\sqrt{27})Kq/L^2$.)
- (d) For a bead with negative charge, $Q < 0$, show that for a small displacement x of the bead from the origin, $|x| \ll L$, that the total force acting on the bead has the mathematical form $F_x(x) \approx c_1x + c_3x^3$, where we are ignoring higher-order powers of x as negligible, and where you should give explicit values for the coefficients c_1 and c_3 .

From this, you can see that for sufficiently small x , $F_x(x) \approx c_1x$ (you can ignore the x^3 term compared to the x term) and so F_x is linear in x , just like the force $-kx$ of an ideal spring about its equilibrium position $x = 0$.

- (e) Use part (d) and Newton's second law of motion $md^2x/dt^2 = \sum F_x$, to show that, for sufficiently small displacements, the bead undergoes periodic harmonic motion, and that the period T of the

motion is given by

$$T = 2\pi \left(\frac{mL^3}{2Kq|Q|} \right)^{1/2}. \quad (1)$$

Discuss briefly whether this expression make physical sense in that the period T varies in a reasonable way as the parameters m , L , q and $|Q|$ are varied (say made bigger).

- (f) Use Eq. (1) to estimate—without using a calculator and without multiplying any digits by hand—the period T in seconds to the nearest power of ten for a Cl^- ion oscillating back and forth between two Na^+ ions in a crystal of table salt NaCl . Assume for this problem that the Na^+ ions are immobile.

Your goal here is to get a sense of the possible order of magnitude of the period of a particular kind of crystal vibration rather than to get a precise answer. The ions in crystalline NaCl lie at the vertices of a cubic lattice, with the spacing between nearest-neighbor ions being approximately $L \approx 6 \times 10^{-10}$ m. I will let you figure out what value of the mass to use in Eq. (1), but do explain your choice and your source.

By finding appropriate data, discuss whether it is reasonable to treat the sodium and chlorine ions as point particles when using Coulomb's law.

- (g) Explain briefly and qualitatively whether you would expect the period T of the Cl^- oscillations to increase, decrease, or stay the same if you include the forces of other nearby ions.

2. A circle of radius R has its center at the origin of an xy -coordinate system that lies in the plane of the circle. The circle has a linear charge distribution λ (lower case Greek lambda) along its circumference that varies as $\lambda(\theta) = \lambda_0 \sin(\theta)$, where $\lambda_0 > 0$ is a positive constant and where θ is the polar angle of a point (x, y) on the circle. Show that the electric field vector \mathbf{E} at the center of the circle is given by

$$\mathbf{E} = - \left(\frac{\pi K \lambda_0}{R} \right) \hat{\mathbf{y}}, \quad (2)$$

and explain briefly and qualitatively why the direction of this vector is reasonable physically.

3. The z axis of an xyz -coordinate system has a positive linear charge density λ . A glass rod of length L is then rubbed with silk to produce a total positive charge Q that is uniformly distributed over the glass rod. The rod is then placed on and parallel to the positive y axis so that the end closest to the origin is a distance d from the z axis. Calculate the total force vector acting on the glass rod in terms of λ , L , Q , and d .

Next estimate the magnitude of the force to the nearest power of ten by assuming that $Q \approx 10^{12}e$ (typical order of magnitude for charging by friction), $L = 30$ cm, and the charge density λ has the same value as that of the glass rod. Would you expect to see a sizable acceleration of the glass rod in the laboratory?

4. From the Knight book and from class, you have learned that the electric field of an infinite plane covered with a uniform (constant) surface charge density σ was shown to have the remarkable property of being independent of the distance to the plane, with the magnitude E at any point in space being given by $E = 2\pi K\sigma = \sigma/(2\epsilon_0)$ where the the constant ϵ_0 (which is called the permittivity constant or “vacuum permittivity” or “permittivity of free space”) is related to Coulomb's constant K by $K = 1/(4\pi\epsilon_0)$.

Derive this result $E = 2\pi K\sigma$ by considering a uniformly charged plane to be the union of many parallel uniformly charged lines. More precisely, consider a charged sheet with surface density σ that is located at the plane $z = 0$ of an xyz -Cartesian coordinate system, and consider a point $\mathbf{X}_0 = (0, 0, z_0)$ sitting a distance z_0 above the plane. Now calculate the electric field $\mathbf{E}(\mathbf{X}_0)$ at this point by decomposing the plane $z = 0$ into infinitely many parallel wires all parallel to the x -axis. Each wire can be labeled

by its y coordinate and the wire at y has infinite length and an infinitesimal width dy , and each has the same linear charge density $\lambda = \sigma dy$. Now use superposition and integration with respect to the y variable to add up the contributions to \mathbf{E} at \mathbf{X}_0 from the infinitely many wires.

5. Consider a three-dimensional xyz Cartesian coordinate system. The plane $z = -H < 0$ is covered with a uniform positive surface charge density $\sigma > 0$, the parallel plane $z = H$ is covered with a negative surface density -2σ , and the x axis is covered with a positive linear charge density $\lambda > 0$. In terms of H , σ , and λ , give mathematical expressions for the electric field vector \mathbf{E} at the three points $(x, y, z) = (0, 1, 0)$, $(0, 1, 2H)$, and $(0, 1, -2H)$.

6. Consider a thin non-conducting hemispherical shell of radius R such that the shell is covered everywhere with a uniform positive surface charge density $\sigma > 0$. Determine the direction of the the electric field vector \mathbf{E} at the center of the base of the hemisphere, and show that its magnitude is $\pi K \sigma$, independent of the radius.

Hint: One way to solve this problem is to decompose the hemisphere into lots of infinitesimal rings each parallel to the base, and then use superposition and integration to sum over the contributions to the electric field of each ring. If you try this approach, it is helpful to label each ring by an azimuthal angle θ for which $\theta = 0$ is the circle at the base and $\theta = \pi/2$ is the top of the hemisphere. Then a particular infinitesimal ring will have radius $r = R \cos(\theta)$, and will have an infinitesimal charge dq equal to the area of the infinitesimal ring times the charge density σ , which you should explain in your homework to be given by $dq = [(2\pi r) \times (Rd\theta)]\sigma = 2\pi\sigma R^2 \cos(\theta) d\theta$. You can now get the answer by using the formula for the electric field along the axis of a charged circular ring, see Eq. (26.17) on page 761 of the Knight 3e book.

Alternatively, if you have studied some multivariable calculus, you can again use superposition but this time use a double integral to add up the contributions to the electric field of all the infinitesimal patches of area $R^2 \cos(\theta) d\theta d\phi$ centered on the angles (θ, ϕ) of a spherical coordinate system, for $0 \leq \theta \leq \pi/2$, and $0 \leq \phi \leq 2\pi$. You should verify that you get the same answer in both cases.

7. Consider an infinitely-long cylinder of radius R_1 that is made of a non-conducting material with uniform volume charge density $\rho_1 > 0$. This cylinder is then covered with a coating of thickness $d = R_2 - R_1$ of a second non-conducting material with uniform charge density $\rho_2 > 0$ to form a new cylinder of radius $R_2 > R_1$ that is coaxial with the original cylinder. The resulting object is thus an infinitely long cylinder of radius R_2 with a coaxial cylindrical core of radius R_1 . (This object is geometrically similar in structure to a fiber optic cable used to transmit information on the Internet, with a cylindrical glass core covered with a plastic outer coating called the “cladding”, see Figure 23.24(b) on page 666 of the Knight book. But the fiber optic cables would be made of neutral materials.) Obtain and give an explicit expression for the electric field vector $\mathbf{E}(P)$, magnitude and direction, at all points of space outside this object.

Some suggestions: use the cylindrical symmetry of this object to deduce the direction of the electric field at any point in space and to argue that the electric field must have the same magnitude at all points that are the same distance r from the axis of this object. Second, use superposition to reduce this problem to the effects of two separate solid cylinders. Finally, use the fact (which we will prove next week using Gauss’s law but that I believe you will accept as plausible) that the electric field at a point outside a uniformly charged infinite cylinder of radius R is the same as the electric field of an infinite straight charged line running along the axis of the cylinder, with some linear charge density λ whose value you need to figure out as part of this problem. One way to verify that your answer is correct is to see if your solution reduces to reasonable answers in the limit $R_1 \rightarrow 0$ (a single cylinder of radius R_2 with charge density ρ_2) and in the separate limit $R_2 \rightarrow R_1$ (a single cylinder of radius R_1 with charge density ρ_1).

I hope that this problem convinces you of the power of using symmetry and superposition: once you know how to get the electric field for one object (here an infinitely long uniformly charged cylinder),

you can solve a variety of other interesting problems right away. Can you also see how to get an answer if the axis of the inner cylinder is parallel to but not coaxial with the axis of the outer cylinder?

8. To the nearest integer, please give the time in hours that it took you to complete this assignment.

9. Optional Challenge Problems

From time to time, I will include challenge problems in the assignments that are substantially harder but hopefully interesting and worthy challenges for some of you. These challenge problems are completely optional, do not feel obliged to try any of them. If you do make some progress in solving one or more of these problems, please write up what you have done and give it to me directly (not to the grader) and you can do this at any time over the semester. While I will not grade your answers nor give you any official credit, any progress you make could have some positive influence on your final course grade by the end of the semester.

I do ask though that you solve these on your own, without getting help from the Internet or from people at Duke such as more senior students, graduate students or professors. Also, please do not work on any of these challenge problems until *after* you have completed the assigned homework problems, it is much more important that you master the basics first.

- (a) Challenge problem 2.1: A long thin rod with end points A and B has a uniform charge density λ . Show that the electric field due to the rod at an arbitrary point C (lying outside the rod) points along the bisector of angle ACB.

This problem has an interesting implication, that the electric field vector at any point above the end point of a semi-infinite uniformly charged line must make an angle of 45° with respect to the horizontal (which is parallel to the charged line).

- (b) Challenge problem 2.2: Given two identical parallel thin rigid rods of length L , separation d and uniform charge densities $\pm\lambda$, find an expression for the magnitude of the total force acting on each rod. Is your expression the same as what you would get if you replaced each rod with a point charge at its center with charge $\pm L\lambda$?

- (c) Optional challenge 2.3: You are given a volume V of a malleable incompressible non-conducting material like clay that has a uniform constant charge density $\rho > 0$ throughout its volume. Determine how to shape the material and place the resulting shape so as to produce the largest possible electric field magnitude E at some point \mathbf{X}_0 of interest.

- (d) Optional challenge 2.4: The square root of 2,

$$\sqrt{2} \approx 1.414213562373095048801688724209698078570\dots \quad (3)$$

is an irrational number which means that its decimal expansion is infinitely long with never repeating groups of digits. This implies that $1 + \sqrt{2}$ is also irrational and so the number

$$s = (1 + \sqrt{2})^{100} \quad (4)$$

is also irrational (can you see why?). Since $1 + \sqrt{2} > 1$, s is a number bigger than one and so can be written as some positive integer followed by a decimal point followed by infinitely many decimal digits. Without using a calculator or computer (just using your mathematical smarts), determine what is the 20th digit of s after the decimal point.

10. Optional Tutorial: Learning How to Use the Computer Mathematics Program *Mathematica*

You can do this problem at any time on your own this semester, it is completely optional and there is nothing to hand in.

For most of this course, symbolic expressions and order-of-magnitude numerical estimates will suffice for the answers. Still, there will be times when you need to compute the numerical value of some possibly complicated expression or get some insight about the graphical behavior of some mathematical expression. I encourage you to take advantage of the program *Mathematica* which is far more capable than any hand-held calculator. Using this program is optional for this course but it can help you to confirm your analytical answers or order-of-magnitude estimates, and it can make your homework assignments more enjoyable to write since, once you have learned some basics, it is easy to mathematically typeset equations with *Mathematica* and to include equations, plots, and discussions all in one nicely formatted *Mathematica* notebook. Occasionally, I will also suggest some optional *Mathematica* calculations in the homework to clarify what is going on.

But a warning: while the first few times you use *Mathematica*, you may become giddy from its enormous power and deep mathematical abilities, *Mathematica* is not an automatic quick way to solve all physics and engineering problems or especially to get insight. *Mathematica* can effortlessly generate 30 pages of intricate algebra and then it is a challenge for the human user to figure out what it all means. And it is way too easy to type something incorrectly in *Mathematica* and so generate many pages of wrong results. (As one example, the letters **ab** in *Mathematica* denote a variable by the name “ab” and does not mean the product of the variable **a** times the variable **b**, which would be written in *Mathematica* with a space between the variables like this: **a b**. So if by accident you leave out a space between two variables, your eye will see the product of the variables while *Mathematica* is using something all together different in its calculations and you end up with an incorrect analysis.). So it is extremely important in 162 and in your other physics and math courses that you first become good at analytical calculations and using physical intuition so you can tell when something has gone wrong with *Mathematica* or with some computer code.

To learn about *Mathematica* at a level appropriate for Physics 162, follow these steps:

- (a) First, install the latest version of *Mathematica* (currently version 10.0.0.2), on your computer. As a Duke student, you can get a free copy for PCs, Macs, and Linux machines here

<https://oit.duke.edu/comp-print/software/license/>

Under the category “Software” select the category “Science and Mathematics”, then select your Platform (operating system).

- (b) After installing *Mathematica*, make sure you can actually run it. Start up *Mathematica* by double-clicking on the *Mathematica* icon. After a few seconds, a “Wolfram *Mathematica* 10” window will appear. Click on the “New Document” button in the upper left of this window to create a so-called notebook window. Inside the notebook window, type “1+2” and then type Shift-Enter to execute the command. (Shift-Enter means type the Enter key while holding down the Shift key.) You should get the answer “3”.

If you have difficulty installing and running *Mathematica*, you can get help from Professor Green-side or from the course TA.

- (c) If you are new to *Mathematica*, go through the online tutorial “Hands-on Start to *Mathematica*” available at this URL:

<http://www.wolfram.com/broadcast/screencasts/handsonstart>

There are many other tutorials here to help you get started.

- (d) Finally, download this *Mathematica* notebook

<http://www.phy.duke.edu/~hsg/162/files/mma-tutorial-schroeder-problem.nb>

say to your Desktop, and then open the file in *Mathematica* by simply double-clicking on the file icon. Once the file is opened in a notebook window, start reading through the tutorial and simply follow the instructions.