

Physics 162: Assignment 1

Made available: Wednesday, January 7, 2013

Due: Friday, January 17, 2013, by 11 pm in the 162 homework box

Before the first discussion section dedicated to problem solving (next Wednesday, Jan 14), please read through all of the problems below and try to solve as many as you can. You can then continue to work on these problems during class with your classmates and get some help from myself and from the course TA, Kevin Holway.

You will learn the most if first you try to get as far as you can on your own, since working in groups or asking for help too early can weaken your problem solving ability. Although it takes patience to struggle with a problem, the creative thinking you carry out as you explore different approaches, including wrong or inefficient ones, is important for gaining skill.

In addition to trying to get as far as you can on your own, try to solve these and future problems by first thinking conceptually and qualitatively, before setting up and solving a problem in mathematical form. Problem 3 below is a good example: you should be able to identify the sign and rough location of the third charge by thinking qualitatively, before doing any mathematics, just by relying on the qualitative observation that the electrical force between two objects becomes weaker with increasing spatial separation.

Finally, please read carefully the part of the course syllabus concerning how you should write up your homework assignments. You will get partial or no credit for your answers, even if they are correct, if you do not follow the guidelines in the syllabus.

Problems

1. Learning How to Use Newton's Generalized Binomial Theorem

The following problem is a guided tutorial, and involves two brief calculations at the end of the discussion for you to hand in as the answers to this problem.

Numerous times during this semester and later on if you have a career that involves physics or engineering, you will be using Isaac Newton's powerful and valuable generalized binomial theorem

$$(1+x)^\alpha \approx 1 + \frac{\alpha}{1} \cdot x + \frac{\alpha}{1} \cdot \frac{\alpha-1}{2} \cdot x^2 + \dots, \quad \text{for } |x| \ll 1, \quad (1)$$

to either simplify analytically (non-numerically) some expression that involves one or more parts raised to powers (which helps you to make quantitative predictions and to get some intuition, see Problems 4 and 6 below), or to obtain quick numerical estimates without need of a calculator.

Eq. (1) holds when the quantity x has a magnitude $|x|$ much less than one, written $|x| \ll 1$. Here the quantity x can be a number or symbolic expression, the symbol $\approx$ means "approximately equal to", and the quantity α (lower case Greek alpha) is some arbitrary nonzero real exponent. The dots ... in Eq. (1), often stated in English as "higher-order terms", refer to the infinitely many further terms in an infinite series that are so small compared to the listed terms (always the case if $|x|$ is taken to be sufficiently small) that they can be ignored. For example, the next term on the right in Eq. (1) is proportional to x^3 , and cubing a tiny number, say 10^{-5} , leads to the much tinier number 10^{-15} , which generally can be ignored.

There are three useful points to keep in mind when applying Eq. (1). First, in many applications, only the first two terms in Eq. (1) are needed, i.e., $(1+x)^\alpha \approx 1 + \alpha x$ to the lowest power of the small quantity x . The quadratic or higher-power terms tend to be important when the first-order term cancels out. For example, in the expression

$$y = (1+x)^{1/3} + (1-x)^{1/3} - 2, \quad (2)$$

where $|x| \ll 1$, using just the first two terms of Eq. (1) would lead to the wrong conclusion that this expression has the value zero and doesn't depend on x , since, to lowest-order, $(1+x)^{1/3} \approx 1 + (1/3)x$, $(1-x)^{1/3} \approx 1 - (1/3)x$ and so y seems to be zero. But if you use the first three terms in Eq. (1), you will see that the quadratic terms do not cancel and that $y \approx -(2/9)x^2$ to the lowest-power of x , as you should verify. More generally, you need to keep using higher-order powers in Eq. (1) until you find the first non-zero power of x .

Second, given some expression, you should look for some part (or parts) of the expression that has the form "something finite added to something small all raised to some power". You can then manipulate this part of the expression into the form Eq. (1) by factoring out the large part of the sum. For example, if you saw an expression of the form

$$(5 + \epsilon^2)^{\sqrt{2}}, \quad (3)$$

and you were told that ϵ is some quantity that is tiny compared to 5, which means that ϵ^2 is also tiny compared to 5, you can transform Eq. (3) into the form Eq. (1) by factoring out the 5 like this:

$$(5 + \epsilon^2)^{\sqrt{2}} = \left[5 \left(1 + \frac{\epsilon^2}{5} \right) \right]^{\sqrt{2}} = 5^{\sqrt{2}} \left(1 + \frac{\epsilon^2}{5} \right)^{\sqrt{2}} \approx 5^{\sqrt{2}} \left(1 + \frac{\sqrt{2}}{5} \epsilon^2 \right), \quad (4)$$

since the power on the right has the form $(1+x)^\alpha$ with $x = \epsilon^2/5$ and $\alpha = \sqrt{2}$. Here I used only the first two terms on the right side of Eq. (1) since there are no cancellations.

A third point is that you should first write powers that appear as denominators as powers in a numerator before applying Eq (1). For example, let's say you want to simplify the mathematical expression

$$\left(\frac{2+y}{3-z} \right)^{1/3}, \quad (5)$$

where you are told that y and z are expressions whose magnitudes are small compared to 2 and 3. You can start the process of applying Eq. (1) by factoring out the "large" quantities like this:

$$\left(\frac{2+y}{3-z} \right)^{1/3} = \left(\frac{2}{3} \right)^{1/3} \left(\frac{1+y/2}{1-z/3} \right)^{1/3}. \quad (6)$$

where the small quantities will be $y/2$ and $-z/3$. You could then write Eq. (6) in the form

$$\left(\frac{2}{3} \right)^{1/3} \frac{(1+y/2)^{1/3}}{(1-z/3)^{1/3}}, \quad (7)$$

and then apply Eq. (1) to the numerator and separately to the denominator, but that will lead to a complicated denominator that is difficult to work with:

$$\left(\frac{2}{3} \right)^{1/3} \frac{1 + (1/3)(y/2)}{1 - (1/3)(z/3)}. \quad (8)$$

Much better is first to convert the denominator in Eq. (6) to a numerator by reversing the sign of the exponent like this

$$\left(\frac{2+y}{3-z} \right)^{1/3} = \left(\frac{2}{3} \right)^{1/3} \left(\frac{1+y/2}{1-z/3} \right)^{1/3} \quad (9)$$

$$= \left(\frac{2}{3} \right)^{1/3} \left(1 + \frac{y}{2} \right)^{1/3} \left(1 - \frac{z}{3} \right)^{-1/3} \quad (10)$$

$$\approx \left(\frac{2}{3} \right)^{1/3} \left(1 + \left(\frac{1}{3} \right) \frac{y}{2} \right) \left(1 + \left(\frac{-1}{3} \right) \left(\frac{-z}{3} \right) \right), \quad (11)$$

$$\approx \left(\frac{2}{3} \right)^{1/3} \left(1 + \frac{1}{6}y + \frac{1}{9}z + \dots \right), \quad (12)$$

where in this last expression I ignored the term proportional to yz since that is a much smaller quantity than x or y itself.

So here are the two things you need to calculate and hand in to see if you understand the above discussion:

- (a) Use Eq. (1) *three* times to show, without using a calculator, that

$$\sqrt{\frac{(1.01)^{7/5} + (0.99)^{7/5}}{2}} \approx 1.000014000, \quad (13)$$

to ten significant digits. Make sure you show clearly how you got your answer without using a calculator, and make sure to explain why the three digits after the 14 are significant.

- (b) Use Eq. (1) to show that

$$6 + \frac{3}{(1 + x + x^2)^{1/5}}, \quad (14)$$

can be accurately approximated by the quadratic polynomial

$$9 - \frac{3}{5}x - \frac{6}{25}x^2 + \dots, \quad (15)$$

for x sufficiently small, $|x| \ll 1$.

2. You have access to an 18th-century laboratory that contains: many identical metal spheres sitting on wood supports that isolate the spheres from a table; a rubber rod and fur that can give the rod a negative charge; a glass rod and silk that can give the glass rod a positive charge; and a long flexible wire connected to an electrical ground. Explain in clear precise steps how to prepare three spheres with charges q_1 , q_2 , and q_3 such that simultaneously $q_1/q_3 = 3/5$ and $q_2/q_3 = -2$.

These spheres could then be used in turn, say, to study Coulomb's law $\mathbf{F} = Kq_1q_2/r^2 \hat{\mathbf{r}}$ for different values of the charges q_1 and q_2 .

3. Two point particles with charges $q_1 = Q > 0$ and $q_2 = 4Q$ are a distance L apart in empty space. These charges are free to move but do not because there is a third charge q nearby in a fixed position.

- What is the magnitude $|q|$ and sign of the third charge, and what is its location with respect to q_1 and q_2 so that this situation is possible? Also discuss why your answer makes sense qualitatively given the relative magnitudes of q_1 and q_2 .
- If the charge q were now made free to move (say you use a laser to burn off a droplet of glue holding it in place), would it move? If so, give the direction of motion.
- Solve this problem again for the case that q_2 now has a negative charge of $q_2 = -4Q$.
- For both cases, discuss whether these are stable equilibria in that *any* tiny change in the location of q_1 or q_2 will leads to a restoring force that pushes q_1 or q_2 back towards their original positions. (Alternatively, an equilibrium is unstable if you can find at least one direction such that a tiny displacement in that direction does not lead to an opposite restoring force.)

4. Three identical point charged masses with mass m and charge q are attached to long threads of length L of negligible mass. These strings are then hung from the same fixed point on a ceiling and, after mechanical equilibrium is achieved, the balls spread out to form a horizontal equilateral triangle with sides of length d .

- (a) Show that the magnitude squared q^2 of the charge can be expressed in terms of the easily measured quantities m , L , and d via this equation:

$$q^2 = \frac{mgd^3}{3KL} \left(1 - \frac{d^2}{3L^2}\right)^{-1/2}, \quad (16)$$

where g is the acceleration due to Earth's gravity in the vicinity of this experiment, and K is Coulomb's constant.

Note: you will want to use somewhere in your analysis the result from plane geometry that the lines connecting vertices to opposite midpoints of any triangle (the so-called three medians of a triangle) intersect at a single point (called the "centroid" of the triangle and which corresponds to its center of mass if the triangle is made of a uniform material like paper) that lies $2/3$ along the distance from the vertex of a median to where the median touches the opposing side.

Eq. (16) is a mouthful of symbols and so here (and later throughout the course, whenever you have a symbolic answer) it is useful to take a little extra time to see if you can determine whether this expression makes physical sense.

Discuss briefly whether the charge's magnitude $|q|$ according to Eq. (16) does behave reasonably in the regime for which d is small, or for the regime for which m is small, or for the regime for which L becomes large (with all other parameters considered fixed).

Note how if the triangle is small in size compared to the string length L , $d \ll L$, so that $-d^2/(3L^2)$ can be ignored compared to 1, then

$$\left(1 - \frac{d^2}{3L^2}\right)^{-1/2} \approx (1 - 0)^{-1/2} \approx 1^{-1/2} \approx 1. \quad (17)$$

Eq. (16) then takes on the simple form

$$|q| = \left(\frac{mgd^3}{3KL}\right)^{1/2} \quad \text{for } d \ll L. \quad (18)$$

- (b) Finally, get some intuition regarding the order of magnitude of this problem by using Eq. (18) to estimate the charge's magnitude $|q|$ in coulombs *to the nearest power of ten* (no significant digits) for the specific values $m = 1$ g, $L = 1$ m, and $d = 5$ mm. Show how you got your answer without a calculator, i.e., how you replaced various numbers by powers of ten and then combined the powers of ten.

To the nearest power of ten, how many electron units e of charge does $|q|$ correspond to? Is this a large or small amount of charge compared to what is produced by rubbing two hand-sized objects like a rod with silk?

Note: watching the 162 video

<http://www.phy.duke.edu/~hsg/162/files/video-powers-of-ten-estimation.mp4>

is a good way to learn how to do quick order-of-magnitude estimates.

5. A cube with side of length L has identical point charges Q located at each of its eight vertices $(0, 0, 0)$, $(0, 0, L)$, $(0, L, 0)$, $(L, 0, 0)$, $(L, L, 0)$, $(L, 0, L)$, $(0, L, L)$, and (L, L, L) . Show that the vector force $\mathbf{F} = (F_x, F_y, F_z)$ acting on the charge at the origin is given by

$$\mathbf{F} = - \left(1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{9}\right) \left(\frac{KQ^2}{L^2}\right) (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \approx -1.90 \left(\frac{KQ^2}{L^2}\right) (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \quad (19)$$

where K is the Coulomb constant and $\hat{\mathbf{i}} = (1, 0, 0)$, $\hat{\mathbf{j}} = (0, 1, 0)$ and $\hat{\mathbf{k}} = (0, 0, 1)$ are the unit vectors along the positive x , y , and z axes respectively.

This kind of calculation would give some insight about how what kind of net force acts on an atom at the corner of a block of an ionic crystal like NaCl.

Hint: reduce the amount of algebra by using the symmetry of the cube.

6. A point charge can exert a net force on a neutral object such as an electric dipole (and, by Newton's third law of motion, a neutral object like a dipole can exert a net force on a point charge). To explore this quantitatively, consider an electric dipole that consists of two point charges $\pm Q$ attached to the ends of a rigid insulating rod of length d . The dipole is centered on the origin of the x -axis such that the charge $Q > 0$ is at location $x = d/2 > 0$ and the charge $-Q$ at location $x = -d/2$. A third point particle with charge $q > 0$ is at positive x coordinate $x = L$ far from the origin so $L \gg d$.

- Deduce and give an exact expression for the net force exerted on the dipole by the charge q .
- Explain qualitatively, without any mathematics, whether the net force on the dipole points towards or away from the positive charge q .
- Use the generalized binomial theorem Eq. (1) to show that the force given by your exact expression has a magnitude approximately equal to the value

$$|F_{\text{net}}| \approx \frac{2Kq(Qd)}{L^3}. \quad (20)$$

The product Qd that appears in the numerator is a measure of the strength of an electric dipole and is defined to be the dipole moment $p = Qd$ with symbol p . Note also the inverse *cube* dependence of Eq. (20): the force on the neutral dipole decreases more rapidly at large distances than an inverse square law.

Note: as mentioned in Problem 1 above, do *not* apply the binomial theorem to expressions in denominators, you just end up with a lot of unhelpful algebra. Instead, rewrite powers so that they appear in a numerator and then use Eq. (1).

7. You have the brilliant insight that gravity might not be needed after all to explain the orbital motion of the Moon around the Earth because the force of attraction is really electrical. Explore your insight by assuming that the Earth and Moon are point particles with opposite charges $Q = Ne$ and $-Q$ so that they attract each other.

- Explain briefly but quantitatively whether it is reasonable to assume that the Moon and Earth are point particles for this problem. (With respect to Coulomb's law, two particles can be treated as point-like if their physical sizes are much smaller than the distance separating them.)
- Estimate to the nearest power of ten the number N of electron charges e needed for the Moon to maintain its current orbit if the only source of attraction is the electrical force. Also give the value of this charge Q in coulombs to the nearest power of ten. (Note how in SI units, a unit corresponding to someone's name is written in lower case "coulomb" while the abbreviated symbol is written in upper case, "C".)

To get some intuition as to whether your answer N is "large" or "small", also give your answer as the fraction (to the nearest power of 10) of N to the total number of electrons on the Moon. Explain any assumptions you need to make to get your estimate.

Note: in nearly all problems in this course, it is best to get the answer fully symbolically before substituting any numbers, then you can see easily how your answer depends on various parameters. Once you have a symbolic answer, round all numbers that appear in the formula right away to the nearest power of ten *before* evaluating the formula, the algebra is then easy and quick since only powers of ten appear.

There is one case though when you don't want to round to the nearest power of ten right away, and that is when a number is raised to an integer power. For example, $2.4^5 \approx 80 \approx 10^2$, compared

with rounding right away in which case you get $2.4^5 \approx 1^5 = 1$, you are two orders of magnitude off!

- (c) Predict an experimental consequence of your “electrical theory” of the Moon’s orbital motion that would allow you to falsify this idea using experiments on Earth.

You can get various physical quantities such as the mass of the Earth, the mass of the Moon, and the average radius of the Moon’s slightly elliptical orbit from the Internet, for example try Wolfram’s Alpha search engine <http://www.wolframalpha.com/>. Whatever numbers you use, please give a reference so I know where you got them.

8. To the nearest integer, please give the time in hours that it took you to complete this assignment, including time working on these problems in class.

Also, if you got help with this assignment, please give the names of the people from whom you got help (classmates, the TA, myself, etc).