Physics 162: Answers to End-of-Class Questions

January 29, 2015

Is it possible/reasonable to approximate large wires and things using Gauss's law? Or would that be too inaccurate?

This question relates to something we already discussed last week and that Knight discussed in Chapter 26, which is that, provided a point of interest is much closer to a wire than the length of the wire (and the point is well away from the ends of the wire) or that the point is much closer to some flat charged area than the width of the area (and again not close to the edges of the charged area), then the electric field at that point is well approximated by the electric field of an infinite wire or of an infinitely wide plane. See page 760 of the Knight book, the paragraph right above the "Stop to Think" question.

It is not possible to apply Gauss's to finite charged wires or to finite flat charged areas because the edges of the wire or of the areas prevent the electric field from having a simple symmetry, which in turn prevents one from finding a mathematical Gaussian bubble such that the electric field has a constant magnitude at all points on the surface and the electric field is perpendicular to or parallel to the surface at all points.

So the idea is to apply Gauss's law to the ideal most symmetric objects, and then use the resulting formulas (say for an infinite line charge or infinite flat plane) to points sufficiently near finite versions of these symmetric objects.

Can you please specify the times when the head of the vector matters?

I am not sure of the context in which you are asking this question and it might be best to discuss this in person, when we can draw pictures for one another.

But let me try to answer the question I think you are asking. Let's say that we have a point particle with charge Q at some known location in space that I can denote by some position vector

$$\mathbf{r}_Q = (x_Q, y_Q, z_Q)\,, \tag{1}$$

with coordinates x_Q, y_Q, and z_Q. You should think of the vector $\mathbf{r}_Q$ as some arrow pointing from the origin of an xyz coordinate system to the point (x_Q, y_Q, z_Q) in space. Now let's say we want to know the electric field produced by Q at some other point P in space:

$$\mathbf{r}_P = (x_P, y_P, z_P)\,. \tag{2}$$

You should think of the position vector $\mathbf{r}_P$ as an arrow pointing from the origin to the point P in space. Then according to Coulomb's law, the electric field at P due to Q is given by

$$\mathbf{E}(P) = \frac{KQ}{d^2}\,\hat{\mathbf{r}}_{Q\to P}, \tag{3}$$

where d is the distance from Q to point P, and where $\hat{\mathbf{r}}_{Q\to P}$ is the dimensionless unit vector that points from the charge Q to the point P.

I hope that, from your knowledge of vector addition and subtraction, you know that the vector

$$\mathbf{r}_{P\to Q} = \mathbf{r}_P - \mathbf{r}_Q = (x_P - x_Q, y_P - y_Q, z_P - z_Q)\,, \tag{4}$$

is the vector that goes from point Q to point P and so the unit vector $\hat{\mathbf{r}}_{Q\to P}$ is given by;

$$\hat{\mathbf{r}}_{Q\to P} = \frac{\mathbf{r}_P - \mathbf{r}_Q}{|\mathbf{r}_P - \mathbf{r}_Q|} \tag{5}$$

$$= \left(\frac{x_P - x_Q}{\sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2}}, \frac{y_P - y_Q}{\text{etc}}, \frac{z_P - z_Q}{\text{etc}} \right), \tag{6}$$

Which is the tip of the vector and which is the tail is determined by the order of subtraction, here P is the tip of the vector and the source charge Q is the tail of the vector.

Why is flux so representative of electric field?

I am not sure what you are asking, is it why one is able to deduce the total charge enclosed in an arbitrary mathematical bubble containing various charges from the flux through that bubble, i.e., why Gauss's theorem is true?

This question would then reduce to how does one prove Gauss's law, or at least how does one show that Gauss's law is plausible. I answer this question partly in the previous end-of-class questions dated January 28. If the answer there is not satisfactory, I am glad to meet with you and and discuss this further.

Is flux a mathematical creation to simplify a calculation or does it have a true physical meaning? What are the implications of flux?

In some contexts, the flux of a vector field through some surface of interest is an actual physical quantity, often of significant importance, while in other circumstances, the flux is just some abstract but still useful number to compute.

For example, fluids like air or water that are in motion can be characterized by a velocity vector field $\mathbf{v}(x, y, z)$ that indicates the velocity $\mathbf{v}$ of the fluid at any point (x, y, z) inside the fluid. If you have some area vector $\mathbf{A} = A\hat{\mathbf{n}}$, and if the fluid has a mass density ρ (that can possibly also vary spatially, at least for gases), then the flux of the fluid through the area A

$$\Phi = (\rho \mathbf{v}) \bullet \mathbf{A}, \tag{7}$$

has the physical meaning of how much fluid mass is passing through the area A each second, that is it tells you something about the rate at which mass is being transported through the area $\mathbf{A}$ via the fluid's motion. So this kind of flux is quite important for any problem that involves fluid motion which includes mechanical engineering, aerospace engineering, physiology, biophysics, chemistry, meteorology, ocean dynamics, plasma physics, astrophysics, and so on.

For almost any quantity that can be transported by fluid motion, there is a corresponding flux: there is a mass flux, energy flux, momentum flux, angular momentum flux, and charge flux. (We will be discussing charge fluxes in about two weeks, when we discuss the motion of charges in wires.)

The flux $\mathbf{E} \bullet \mathbf{A}$ of an electric field through some given area $\mathbf{A}$ and the corresponding magnetic flux $\mathbf{B} \bullet \mathbf{A}$ that we will discuss later this semester, have no known physical interpretation in terms of the transport of some quantity via fluid motion. By analogy to Eq. (7), we could try interpreting the electric field $\mathbf{E}$ as a velocity field with mass density $\rho = 1$, but no one has found any quantity that is "moving" along with the electric field. So electric and magnetic fluxes end up being abstract quantities that are natural to consider by analogy to fluxes in fluids, and that turn out to be highly useful because these fluxes turn out to be deeply related to charges and currents.

It is not uncommon in physics that certain abstract quantities turn out to be highly valuable to measure, compute, or understand. An example of particular importance is the complex-valued wave function $\Psi(x, y, z, t)$ that characterizes the quantum mechanical properties of some atomic or molecular system. This is the key quantity in the quantum theory of matter and yet it has do direct physical meaning itself (although Ψ can be used to deduce some physical results).

Why does the Gaussian surface have to be symmetric when calculating the total flux $\Phi = \oint \mathbf{E} \bullet d\mathbf{A}$ in Gauss's law $\Phi = Q_{\mathrm{encl}}/\epsilon_0$?

Gauss's law is a completely general relation that links the total electric flux through a mathematical bubble to the total charge enclosed by the bubble. But it turns out that the electric field **E** produced by highly symmetric continuous arrangements of charge (a ball with spherical symmetry, an infinitely long cylinder with translational and rotational symmetry, and an infinitely big plane that has translational and rotational symmetries) is such that one can find a particular Gaussian bubble such that two things hold simultaneously:

1. At every point on the bubble's surface, the electric field **E** is either perpendicular to or parallel to the surface of the bubble.
2. The magnitude E of the electric field is everywhere constant over the surface.

Under these rare conditions, the electric flux integral becomes a product of the total surface area A of the bubble times the constant unknown value E of the electric field magnitude

$$\oint \mathbf{E} \bullet d\mathbf{A} = AE, \tag{8}$$

and then one can solve for E if one knows the total charge inside the bubble.

What is the difference between calculating the flux inside or outside of the sphere/shell/?

I am not sure I understand your question, but there is no technical difference when calculating the electric flux through some spherical Gaussian bubble that lies inside some physical object or outside the same object. In both cases, the electric flux $\mathbf{E} \bullet d\mathbf{A}$ over the surface of the bubble—the thing on the left side of Gauss's law—has the simple form $(4\pi d^2) \times E$ where d is the radius of the Gaussian bubble (not the radius of the physical charged ball). In both cases, you have to add up all the charge enclosed Q_{encl} in the Gaussian bubble sphere of radius d. Equating the left side to the right side then let's you solve for the unknown electric field magnitude $E(d)$ at a point a distance d from the center of the spherical ball of charge.

For an irregular but still closed surface, is Gauss's law still applicable? What "bubble" would you use?

Gauss's law is true for any mathematical bubble of any shape (ok, it can't be too pathological mathematically so it needs to be reasonably smooth) and it just doesn't matter how messy are the charges inside the bubble, whether they are some arrangement of point charges or some sphere-like thing or some huge mess of charged objects of different shapes, or even some infinitely wrinkled fractal arrangement of charges.

But if you want to use Gauss's law to calculate the electric field of some arrangement of charges, then there are just three cases for which this works: a spherically symmetric ball of charge, a cylindrically symmetric infinite rod of charge, and an infinitely wide slab of charge that that is everywhere the same in space as one moves about on the surface of the slab. For all other charged objects (cube, finite cylinder, finite planar area, cone, ellipsoid, torus, etc), there are no mathematical bubbles that enclose the object such that the electric field is everywhere perpendicular or parallel to the bubble and such that the electric field magnitude E is constant over the surface of the bubble. For all of these other cases, you would have to get the electric field by superposition and integration, as we did in Chapter 26.

Were the cylinder, sphere, etc symmetry arguments the beginnings of group theory?

These arguments are somewhat related to group theory, but these problems are a bit too simple to require using group theory in any serious way.

Given any physical object of any kind (sphere, cylinder, infinite cylinder, cone, torus, a cloud), one can ask what kinds of transformations one can apply to the object such that the object does not change after the transformation. For example, one can rotate a sphere about any axis passing through its center or reflect the sphere through any plane containing its center and you end up with the exact same set of points as before, the same sphere. For an infinite cylinder, one can rotate the cylinder by an arbitrary angle about its central axis, one can reflect the cylinder in any plane perpendicular to its central axis, and one can translate the cylinder by some finite distance along its central axis, and you end up with the same set of points as before. There is a mathematical way to describe these operations and one can show that the set of such operations indeed forms a mathematical group.

Then one would mathematically say that Gauss's law can only be used to calculate the electric field of some object when that object is invariant or symmetric (remains physically unchanged) when all the elements of some particular group are applied to that object.

Is it possible for a three-dimensional object to have a volume charge density? If the object is a conductor, all the charge will move to the surface of the object, and if it is not, how can you reach inside the object without messing with the other charges in the object?

This is a really good practical question: is it possible to create three-dimensional charge densities $\rho(x, y, z)$ and, if so, how?

There are some trivial ways to do this. For example, it is quite possible to create little plastic cubic centimeter blocks that each have one charged particle trapped inside (and physicists know how to make excellent particle traps that can hold a charged particle like an electron or positron basically forever in vacuum at some desired point in space). And then it would be trivial to make your three-dimensional charge density by just physically stacking the blocks how you like. Or using slightly more modern technology, you could create a three dimensional charge density by using a 3d printer in which you add charges occasionally to the plastic drops being created by the printer.

This would be a macroscopic charge density and you might say that this is cheating, you want a charge density created at the molecular or atomic scale. This too can be achieved. One way would be to use so-called molecular beam epitaxy (which should win its inventors a Nobel prize at some point). The idea is to create a high quality vacuuum, and then carefully spray one layer at a time of atoms or ions of different kinds onto some substrate, creating a three-dimensional structure one atomic layer at a time. Or one can use a device called an atomic force microscope to pick up and deposit one atom at a time in desired locations on some surface. Either way, you can alternate layers of non-conducting materials with layers of charges in such a way to produce pretty much any desired three-dimensional charge density you like. (It might be expensive and time consuming to do this, but we know how to do it.)

Having said this, I need to admit that I don't know any examples of artificial solid three-dimensional charge densities created like this, and the examples I have been discussing in class and that Knight has been discussing in his book are indeed rather idealized impractical and largely pedagogical examples that illustrate how to use Gauss's law.

Where one does find three-dimensional charge densities of considerable importance are in beams of charged particles, e.g, the millimeter-diameter proton beam in the Large Hadron Collider or the electron beams used to etch integrated circuits by lithography. These are not truly one-dimensional beams, they are three-dimensional cylindrical-like arrangements of moving charged particles and it is quite important to understand the electric and magnetic fields produced by these beams since they affect the quality and stability of the beams.

Are there applications of multidimensional fluxes?

There are many many examples of multidimensional fluxes, both scientific and mathematical. Basically, anytime you have a vector field of any kind, and vector fields show up in many areas of science and math, no matter what the dimension of the vectors, it is often useful to ask what is the flux of that vector field through some higher-dimensional area.

Let me give you one rather neat example, which involves how to define "friction" mathematically and generally. You have hopefully seen the mathematical equation for a damped harmonic oscillator:

$$m\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + kx = 0, \tag{9}$$

where the term with the α coefficient (the "damping term") is a simple although crude way to model friction. This so-called ordinary differential equation determines how the position $x(t)$ of some point particle attached to a spring of spring constant k evolves over time.

But there are many mathematical models (sets of differential equations) that have been invented that have no physical context, e.g., economic models, models of population growth, models of information flowing through the Internet, models of abstract neurons dynamically interacting in some intricate network, and so on. It turns out that it is possible to identify whether these models have some kind of friction and it involves an abstract high-dimensional flux.

Briefly, there is a lemma that states that any set of differential equations of any order can be rewritten to have the form of a vector set of first-order differential equations like this:

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}), \tag{10}$$

where the solution vector $\mathbf{u}(t)$ is some N-dimensional vector that looks like this:

$$\mathbf{u} = \left(u_1(t), u_2(t), \ldots, u_N(t)\right), \tag{11}$$

and where the right side of Eq. (10) is a N-dimensional vector function of $\mathbf{u}$ that looks like this;

$$\mathbf{f}(\mathbf{u}) = \left(f_1(\mathbf{u}), f_2(\mathbf{u}), \cdots, f_N(\mathbf{u})\right), \tag{12}$$

where the N functions are known (specified) functions of the $\mathbf{u}$. For example, the second-order differential equation Eq. (9) can be written in this 2-dimensional first-order form:

$$\frac{d(u_1)}{dt} = f_1\left(u_1, u_2\right) = u_2, \tag{13}$$

$$\frac{d(u_2)}{dt} = f_2\left(u_1, u_2\right) = -\frac{\alpha}{m}u_2 - \frac{k}{m}u_1, \tag{14}$$

Then it turns out that the flux of the vector field $\mathbf{f}(\mathbf{u})$ in the N-dimensional space through some $(N-1)$-dimensional tube-like surface that encloses the curve $\mathbf{u}(t)$ traced out by the solution is negative if the system has friction, zero if the system has no friction, and is positive if the system as a kind of "anti-friction" that would correspond to an unphysical steady increase in the energy of the system.

So studying fluxes in abstract N-dimensional spaces defined by mathematical models of differential equations turns out to be a valuable strategy for understanding the properties of solutions of such models, even without finding the solutions explicitly. You can learn more about this beautiful and powerful geometric way of thinking about mathematical models by taking the course Physics 513 on nonlinear dynamics.