Physics 162: Answers to End-of-Class Questions

January 23, 2015

How can you tell where the maximum electric field magnitude E is away from a ring/cylinder/etc?

At the level of Physics 162, there are two ways. One is conceptual, looking for implications of symmetry and of the inverse-square behavior of Coulomb's law. An example would be the electric field on the axis of a charged ring or charged cylindrical surface: we know from symmetry that $\mathbf{E} = \mathbf{0}$ at the center of the ring or cylinder (since all electric field vectors cancel), and we know the electric field dies off as $1/r^2$ far from these charged objects. So that implies there has to be least one maximum in E between the center and far from from the object.

The other approach at the level of 162 would be to find an explicit mathematical expression for the electric field $\mathbf{E}(x, y, z) = (E_x(x, y, z), E_y(x, y, z), E_z(x, y, z))$, which then leads to an explicit expression for the magnitude $E(x, y, z) = |\mathbf{E}| = \sqrt{E_x^2 + E_y^2 + E_z^2}$ of the electric field as a function of position (x, y, z). You could then use calculus (more generally, multivariable calculus) to find where $E(x, y, z)$ had extrema (minima or maxima) as a function of position.

For many problems in 162, the electric field strength ends up being a function of one variable (say distance x along some axis, angle θ along some circular arc) and then you would then set the derivative to zero, $dE/dx = 0$ to locate the values of x corresponding to an extremum. You could then use the physics context to deduce whether an extremum was a maximum or minimum, or, with more effort, you could calculate and look at the sign of the second derivative d^2E/dx^2 to deduce mathematically which extrema are maxima. (I can't think of any cases this semester where you will need to calculate second derivatives.)

For the more general case when the electric field magnitude $E = E(x, y, z)$ depends on several spatial coordinates, you could find extrema by setting the so-called gradient of E to zero: $\nabla E = (\partial_x E, \partial_y E, \partial_z E) = \mathbf{0}$. This is a more difficult mathematical problem since you now need generally to solve three nonlinear equations for possible locations (x^*, y^*, z^*) of minima. Finding solutions of nonlinear equations can be a very hard problem analytically and numerically, in fact even knowing if solutions exist can be quite challenging. (Google the Clay Mathematics Millenium problems, one of which concerns understanding the properties of the nonlinear partial differential equations that describe fluid dynamics.)

Beyond Physics 162, scientists and engineers would typically write a computer code that can find numerically an approximate value of $E(x, y, z)$ at any point in space, and then one would use a numerical optimization software program to locate approximate possible maxima. Numerical optimization is a subtle and advanced branch of mathematics and computer science, and it can take a lot of effort to find an extremum, especially a global "best" maximum.

Other than capacitors, what are charged parallel plates used for?

Capacitors are by far the most important application of oppositely charged parallel plates since capacitors can store and release electrical energy (search the internet for the concept of "supercapacitors") and they are essential components in many electrical circuits, e.g., for generating oscillations or for measuring time (we will discuss some of these applications later this semester).

The other wide use of charged parallel plates is to steer the direction of a beam of charged particles, and this need shows up in many basic and applied areas of science. See for example Example 26.9 on page 769 of the Knight text and problem 68 on page 779 of the Knight book. When the class tours the TUNL laboratory after spring break, you will see several examples of parallel plates being used to direct a beam of nuclear particles towards some target to do nuclear physics research.

Could you recommend resources to review calculus and trigonometry necessary for this course?

That might be best to do in person, so I can get an idea of what kind of math background you have, so please make an appointment to see me if you want to follow up on your question.

For 162 and future science and engineering courses you take, you should have a small library of math books at your finger tips so you can look up topics as you need them. For example, you should definitely own a hard copy of an introductory calculus text so that you can review topics from time to time as you come across various concepts in your science and engineering courses. (I don't have a particular calculus book to recommend, the one used at Duke in its 111 and 112 math courses should be fine.) You may also want a precalculus textbook that discusses trigonometry, probability theory, and qualitative functions at an appropriate level, but you will need to search the Internet (especially high schools) to find appropriate books.

One book you might like to take a look at is "Basic Training in Mathematics: A Fitness Program for Science Students" by R. Shankar. This physics professor at Yale wrote this book to help a variety of students bring their math skills up to the college level, especially those majoring in physics, biophysics, or engineering.

How does the machine you described work, that can select single electrons and place them at will? What technology allows that precision?

Look up the Wikipedia article "Atomic force microscopy" as one way to learn about this remarkable technology, whose invention led to a Physics Nobel Prize in 1986.

The idea is simple but the implementation (at least the first time around) was difficult. I mentioned in class that there are certain crystals like quartz that have the property of being "piezoelectric", if you squeeze them, an electric field is generated. The opposite also happens, if you apply an electric field (e.g., by putting the crystal between two parallel oppositely charged plates), the crystal can contract or expand by a small amount that is roughly proportional to the strength of the electric field. So the idea is to build a hierarchical device, that has a big piezoelectric crystal to move things a large amount, and then attach a smaller piezoelectric crystal to the big crystal for finer control, and so on. With some effort, you end up with a device that can position the tip of an ultra-sharp needle (just a few atoms wide) anywhere you want in space, by just telling a computer how much electrical field to apply to the various crystals. There are then protocols that people have worked out to pick up and put down atoms or electrons at will, say on the surface of a crystal.

The same technology allows remarkable images to be created of surfaces with sub-atomic resolution. Look over this gallery of images

`http://researcher.watson.ibm.com/researcher/view_group.php?id=4245`

of IBM researchers showing off images they created by placing atoms one by one on a surface and then using an AFM to generate an image of their artwork. One of my favorites is "carbon monoxide man":

`http://researcher.watson.ibm.com/researcher/files/us-flinte/stm5.jpg`

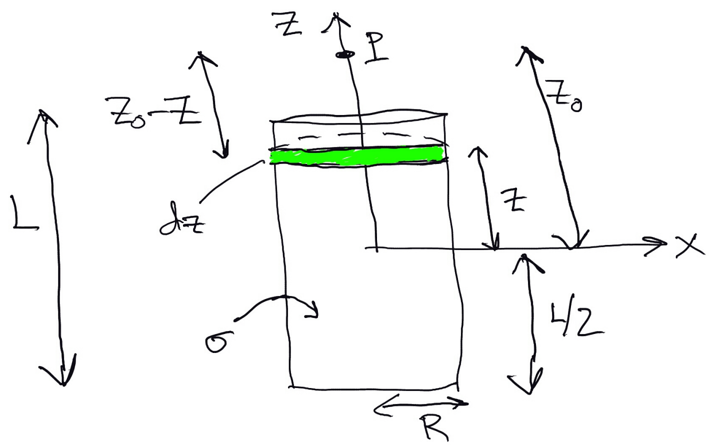
How do we know which variables to switch? Do we just practice until we start having intuition or are there patterns to make it easier?

There are only a few cases you will need to consider over the semester, so you will know how to change variables in most integrals this semester after seeing several examples. See also the answer to the next question, where I give some further information about changing variables to "push" an integral towards some useful form.

The key idea is to change your variable of integration to some dimensionless variable in such a way that no dimensional parameters remain in the integral. This transforms your integral into a dimensionless integral—a pure number that often has a magnitude that varies between 0 and 1 to the nearest power of ten—times some algebraic combination of physical parameters that represent not only the physical units of the answer but indicate the dominant way that various parameters enter into the problem, e.g, some radius might show up as $1/R^3$ or the answer is proportional to some charge squared, Q^2.

Can you explain the calculation of the electric field on the axis of a cylindrical surface, including how to change variables in the integral?

Consider a cylindrical surface of radius R, length L, and uniform surface charge density $\sigma > 0$:



We want to use superposition and integration to calculate the electric field $\mathbf{E} = E_z(z)\hat{\mathbf{z}}$ at any point on the axis of the cylinder. (Because of the circular symmetry of the charged surface about the axis, the electric field can only be parallel to the z axis.) One could of course try to calculate the electric field at any general point in space, not just on the axis, but the resulting integrals (which I encourage you to explore) are not easy to evaluate in any useful form and many researchers would just use a computer program to get the answer.

We solve this problem by thinking of the cylindrical surface as the union of many parallel rings of radius R and of infinitesimal thickness dz, of which a representative ring has been shaded in the above figure. We then get the $\mathbf{E}$ field at some point P by adding up the electric field at P created by each ring.

We need a way to label each infinitesimal ring so that we can include its contribution to the overall electric field at P. One way to do this is to introduce an $x - z$ coordinate system with the origin at the center of the cylinder (it is often useful to put the origin of coordinate systems at a point of high symmetry), and let the z axis be the axis of the cylinder as shown in the above diagram. We then partition the cylindrical surface into identical parallel rings that are really very short cylindrical shells of radius R and length dz.

Let the point P lie at coordinate z_0 on the axis, so $z_0 = 0$ is the center of the cylinder and $z_0 = \pm L/2$ will be parallel with the top and bottom of the cylinder. Let us consider a representative general infinitesimal ring that lies at coordinate z, with $-L/2 \le z \le L/2$ (this is the shaded ring in the drawing). Then we know from the Knight book (see page 761) and from our discussion in class that this ring contributes an electric field at P given by:

$$E_z = \frac{KQ(z_0 - z)}{\left((z_0 - z)^2 + R^2\right)^{3/2}}, \tag{1}$$

where I have used $z_0 - z$ in the formula on page 761 of Knight because $z_0 - z$ is the distance of the point P to the plane of the ring with coordinate z. The charge Q is the total charge on the ring and we get this by multiplying the surface area of the infinitesimal ring by the surface charge density:

$$Q = \text{area} \times \sigma = [(2\pi R)dz] \times \sigma = 2\pi\sigma R dz. \tag{2}$$

I have used the fact that the surface area of a cylinder of length L and radius R is the circumference multiplied by the length, $(2\pi R)L$, and here the cylinder length is $L = dz$.

We now substitute Eq. (2) into Eq. (1) and add up all the contributions by integration, giving us the following one-dimensional integral over the coordinate z:

$$E_z(z_0) = \int_{-L/2}^{L/2} \frac{K\left([(2\pi R)dz]\sigma\right)(z_0 - z)}{\left((z_0 - z)^2 + R^2\right)^{3/2}} \tag{3}$$

$$= K(2\pi R\sigma) \int_{-L/2}^{L/2} \frac{(z_0 - z)\ dz}{\left((z_0 - z)^2 + R^2\right)^{3/2}}. \tag{4}$$

Before trying to evaluate the integral, it is useful to simplify it in several steps. (This is somewhat overkill for such a simple integral, but many integrals are not simple to evaluate and then the simplications are insightful and useful.) A first step is to replace the difference $z_0 - z$ with a single variable, say u, since many integral tables assume integrands of the form $u^2 + c^2$ for some constant c. So let's transform the integral Eq. (4) to the new variable u by writing

$$u = z_0 - z \quad \Rightarrow \quad du = -dz, \tag{5}$$

and we also have to transform the upper and lower bounds to the new variable:

$$L/2 \quad \rightarrow \quad z_0 - L/2, \qquad -L/2 \quad \rightarrow \quad z_0 + L/2. \tag{6}$$

The integral Eq. (4) then becomes (please verify):

$$E_z = K(2\pi R\sigma) \int_{z_0 - L/2}^{z_0 + L/2} \frac{u\,du}{(u^2 + R^2)^{3/2}}. \tag{7}$$

At this point, the integrand is simple enough that we could integrate it directly since the numerator $u\,du$ is proportional to the differential of the quantity $u^2 + R^2$ in the denominator, i.e, $d(u^2 + R^2) = 2udu$, which leads to a particularly easy class of integrals to evaluate, see Eq. (16) below. (If the numerator were not proportional to the differential of $u^2 + R^2$ we would then try using trigonometric substitution to evaluate the integral, e.g., try the substitution $u = R\tan(\theta)$.) But let's make one more simplification of the integral, which is to transform to a dimensionless integration variable w, which will leave an integral that has no physical dimensions. This step often clarifies an integral by writing it as an algebraic combination of physical parameters times a dimensionless integral that then represents some pure number. In many cases, the resulting dimensionless integral has a magnitude of order one (to the nearest power of ten) because its the physical parameters that determine the overall magnitude of some expression and these have been separated from the integral.

So let's define a new integration variable:

$$u = Rw. \tag{8}$$

Since u has units of length and R has units of length, $w = u/R$ is indeed dimensionless. We again change integration bounds to the new variable like this:

$$z_0 + L/2 \quad \rightarrow \quad \frac{z_0 + L/2}{R}, \qquad z_0 - L/2 \quad \rightarrow \quad \frac{z_0 - L/2}{R}, \tag{9}$$

and observe that $u = Rw$ implies $du = R\,dw$. Then Eq. (7) becomes (please verify)

$$E_z = 2\pi K\sigma \int_{(z_0 - L/2)/R}^{(z_0 + L/2)/R} \frac{w}{(w^2 + 1)^{3/2}}\,dw. \tag{10}$$

Eq. (10) represents several clarifications about what is going on:

1. All the physical units have now been collected together in the product $2\pi K\sigma$, which multiplies a non-dimensional (purely numerical) integral. We can see that this combination of physical constants has the correct physical units of electric field units or N/C. To see this, note that σ has units of charge per unit area or $\mathrm{C/L^2}$ where C is the Coulomb unit and L denotes a length. Then $K\sigma$ has the same units as KC/L^2 which indeed looks like KQ/d^2 which we recognize as Coulomb's force law divided by a charge, F/q, which is indeed the units of electric field.

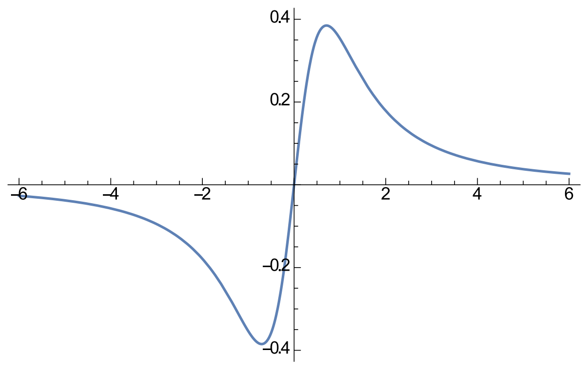
   If you are alert, you may have also noticed that the expression $2\pi K\sigma$ is none other than the electric field strength produced by an infinite plane with uniform surface charged density σ. This is intriguing, it suggests that perhaps we should think of a cylindrical surface as first acting most like a plane and that its finite length L and radius R are "corrections" to the electric field of a plane.

2. Note how the radius R and length L of the cylinder do *not* show up in the product $2\pi K\sigma$ of physical units that give the overall physical units of this expression. This implies that the answer does not depend strongly on ("scale with") the radius or length of the cylinder, e.g., there is not a simple change if we double the cylinder radius or double the cylinder length. Indeed, the integral on the right side of Eq. (10) tells us that R and L only enter via a certain ratio: how big $z_0 \pm L/2$ are in units of R.

3. The integrand in Eq. (10)

   $$\frac{w}{(w^2+1)^{3/2}}, \tag{11}$$

   has no parameters since we have eliminated them all by changes of variables. This means that this integrand is particularly easy to plot and visualize, and to estimate numerically since we don't have to assume arbitrary values for R or L to generate a plot. The function has this form

   and the integral Eq. (10) then corresponds to calculating the area under this curve over some range defined by the integration bounds. From this numerical plot, it would not be hard to make rough estimates, say to one significant digit, of the value of the integral by just estimating the appropriate area under the curve between the two bounds $(z_0 \pm L/2)/R$. In particular, the area under Eq. (11) between 0 and ∞ is easily seen to be 1, so this integral will have a value of at most 1 (consistent with my claim that dimensionless integrals tend to be of order 1) and possibly be as small as zero.

   As one example, when the length L gets large compared to z_0, $L \gg z_0$, the bounds $(z_0 \pm L/2)/R$ approximately become $\pm L/(2R)$, and so the integral must nearly vanish since we are integrating an odd function between nearly nearly equal positive and negative bounds. This makes sense: for $L \gg z_0$, the point P lies deep inside the long cylindrical surface and so the electric field must be tiny at P since there are all kinds of cancelations from the $\mathbf{E}$ fields produced by rings on one size of z_0 with the fields produced by rings on the other side.

   We also see that in the limit $L \to \infty$ corresponding to an infinitely long cylindrical surface, the integration bounds becomes $\pm\infty$, the integral vanishes exactly, and we conclude that the electric field vanishes everywhere on the axis of the cylinder, which also seems reasonable since, at any point P on the axis of an infinite cylinder, the electric field produced by one infinite half of the cylinder is canceled exactly by the field produced by the other infinite half. We will prove this result rather trivially using Gauss's law next week.

Finally, from Eq. (10), we see that as the cylindrical radius R becomes quite large for fixed L and z_0, the bounds $(z_0 \pm L/2)/R$ become tiny numbers, i.e., the integral corresponds to taking the area under the curve plotted above over a range that is very close to zero which is where the integrand itself is going to zero, which implies that the electric field must be tiny for large R. This also makes sense: if the cylindrical surface is becoming very far away (large radius for fixed height), the electric fields from each infinitesimal piece on the surface are becoming weak with distance. Alternatively, a cylinder with large R and fixed L looks an awful lot like a really big ring and we know that the electric field at the center of a ring is zero by symmetry.

I mention all these deductions to illustrate how much one can figure out what is going on physically before one obtains an explicit expression for the integral. This is the most common case in physics and engineering: often one ends up with some complicated integral for which one cannot find an antiderivative in terms of convenient functions. (There is a mathematical theorem that "most" integrands, in a certain measure-theoretic sense, have the property that they can not be expressed as an algebraic expression involving "elementary functions" like polynomials, trig functions, and exponentials.) So the above qualitative reasoning is valuable when trying to confirm that some expression makes sense or that some computer code (which is likely long and complicated) is generating correct results.

OK, we can finally evaluate Eq. (10) and get a final explicit answer for the electric field that we can further analyze, say using Mathematica. We observed that the numerator $w\,dw$ is proportional to the differential of the thing, w^2+1, being raised to a power. This suggests one last change of variables to simplify the integral:

$$s = w^2 + 1 \quad \Rightarrow \quad ds = 2w\,dw \quad \Rightarrow \quad w\,dw = (1/2)\,ds. \tag{12}$$

The integration bounds change one last time:

$$w_2 = \frac{z_0 + L/2}{R} \quad \rightarrow \quad s_2 = 1 + \left(\frac{z_0 + L/2}{R}\right)^2, \tag{13}$$

$$w_1 = \frac{z_0 - L/2}{R} \quad \rightarrow \quad s_1 = 1 + \left(\frac{z_0 - L/2}{R}\right)^2, \tag{14}$$

and so the integral Eq. (10) becomes (please verify)

$$E_z = 2\pi K\sigma \int_{(z_0-L/2)/R}^{(z_0+L/2)/R} \frac{w\,dw}{(w^2+1)^{3/2}} \tag{15}$$

$$= 2\pi K\sigma \int_{s_1}^{s_2} \frac{(1/2)\,ds}{s^{3/2}} \tag{16}$$

$$= \pi K\sigma \int_{s_1}^{s_2} s^{-3/2}\,ds \tag{17}$$

$$= \pi K\sigma \left[\frac{s^{-(3/2)+1}}{-(3/2)+1}\right]\Bigg|_{s_1}^{s_2} \tag{18}$$

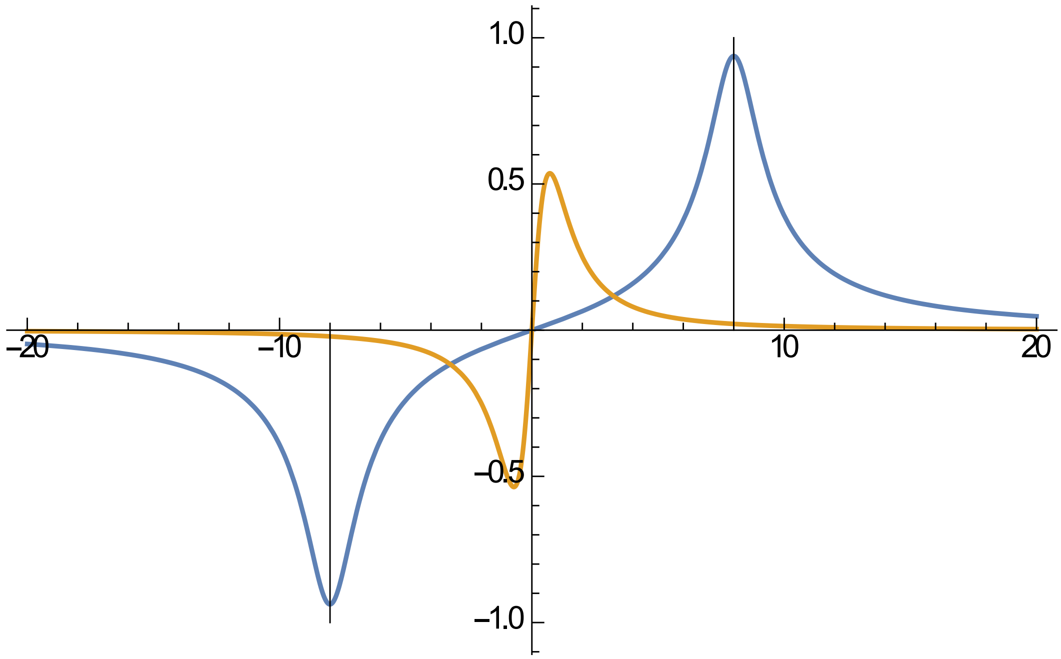
$$= 2\pi K\sigma \left[s_1^{-1/2} - s_2^{-1/2}\right] \tag{19}$$

$$= 2\pi K\sigma \left(\left[1 + \left(\frac{z_0 - L/2}{R}\right)^2\right]^{-1/2} - \left[1 + \left(\frac{z_0 + L/2}{R}\right)^2\right]^{-1/2} \right), \tag{20}$$

and Eq. (20) is our final explicit answer for the electric field $E_z(z)$ at any point z on the axis of a cylindrical shell.

It would be a good exercise for you to practice your skills with the generalized binomial theorem and verify that the answer Eq. (20) reduces to the electric field of a point charge of total charge $Q = [(2\pi R)L]\sigma$ when $z_0 \gg L, R$, and to verify that Eq. (20) reduces to the expression of the electric field of a single ring in the regime $L \ll R$.

To get some understanding of how the electric field of a cylindrical surface differs from that of a ring, I used Mathematica to plot Eq. (20) as a function of z_0 for $R = 1$, $L = 16$ (cylinder is 8 times longer than its diameter), and ignoring the coefficient $2\pi K\sigma$ which only influences the magnitude, but not the shape, of the function:



At the same time, I show the plot Eq. (20) for $R = 1$ and for $L = 0.2$ which approximates the electric field of a ring, which you can think of as a short cylinder; I also multiplied the ring curve by a factor of 7 so that it is easier to see on the same plot, otherwise it will be much smaller in magnitude. You can see directly how the electric field on the axis of a ring differs functionally from the electric field of a cylindrical surface with the same radius. The location of the maximum electric field seem to occur at $z = \pm L/2$ (can you determine mathematically whether or not it occurs exactly at $\pm L/2$?), and the electric field is decreased in magnitude inside the cylinder. The plot also suggests that the asymptotic $1/z^2$ behavior of the electric field for large z, expected when one is supposedly "far" from the cylinder so that the entire cylindrical surface acts like a small point charge, appears surprisingly close to the ends of the cylindrical charge, here for $|z| > 12 \approx 1.5(L/2)$.

This plot supports the qualitative conclusions that many of you reached in class, about what the form of the electric field along the axis of a cylindrical shell: $E_z(z)$ indeed qualitatively has the same form as that of a ring but stretched out horizontally and bigger in magnitude, and with the maxima now occurring at the ends of the cylindrical surface. One detail that is rather clear from the plot but not clear from qualitative reasoning nor immediately from the analytic answer Eq. (20) is that the maximum electric field magnitude seems to coincide rather precisely (perhaps exactly) with the ends of the cylindrical surface which are indicated by the vertical lines at $z = \pm L/2 = \pm 8$.