Course Outline

Lecture 1 : Ising model and Mean field theory (MFT)

Lecture 2 : MFT (continued).
Critical exponents from MFT.

Lecture 3 : Landau Ginzburg approach
$\longrightarrow$ Introducing path integrals.

Lecture 4 : The Gaussian path integral

Lecture 5 : Two-point correlation function
Breakdown of MFT.

Lecture 6 : Renormalization Group. – the big idea.

Lecture 7 : RG for the ϕ^4 theory.

Lecture # 1

Thursday, Feb 27, 2020

ISING MODEL

- spins ± 1 at each site
- Purely classical model.

$$E = -J \sum_{\langle ij \rangle} s_i s_j - h \sum s_i$$

internal microscopic interactions of the system.

external field

To introduce some basic terminology, we ask the question:

Q: What is the ground state of the system?

A Set $h=0$ for now.

a) $J > 0$: clearly, the spins like to be aligned.

Ferromagnetic system

b) $J < 0$: Spins like to be anti aligned

Antiferromagnetic system

But that is just the ground state of the system.
We are interested in the system at a finite temperature T.

	Low temp	High temp.
$J > 0$	spins aligned	Fluctuations become large
$J < 0$	spins antialigned	
	ordered states	Disordered state.

We now need to answer the question

Q: Is there a phase transition at some $T = T_c$?

In a canonical ensemble, the prob. of a conf of energy E is given by

$$P(E) = \underbrace{\frac{1}{Z}}_{\text{normalization}} e^{-\beta E} \quad \text{for temp} \quad \beta = 1/kT.$$

We can use Z to compute everything

$$Z = \sum_{C} e^{-\beta E[C]} \quad \text{where} \quad C : \text{configuration of all the spins} = [S_i]$$

called the partition function.

Two interesting quantities

1) Free energy

$-\beta F(\beta)$ thermodynamic free energy.

$$Z = e$$

or $$F = -E + TS$$

2) magnetisation

$$m = \frac{1}{N} \sum s_i$$ average value of the spin.

$$Z = \sum e^{-\beta E}$$

Note that $$E = -J \sum s_i s_j - h \sum s_i$$

$$\frac{\partial E}{\partial h} = -\sum s_i$$

So $$\frac{1}{Z}\frac{\partial Z}{\partial h} = \frac{1}{Z} \sum \left(-\beta \frac{\partial E}{\partial h}\right) e^{-\beta E}$$

$$= \frac{1}{Z} \sum +\left(\beta \frac{\sum s_i}{N}\right) e^{-\beta E} \cdot N$$

$$\frac{1}{Z}\frac{\partial Z}{\partial h} = \beta N m$$

$$\Rightarrow \quad m = \frac{1}{\beta N} \log Z.$$

Our task: compute the partition function Z.

We will not compute it directly.

(It can be done $d=1$ and $d=2$).

We will make some arguments that will apply more broadly.

Effective free energy

$$Z = \sum_{m} \sum_{\{s_i\}|m} e^{-\beta F[s_i]} = \sum_{m} e^{-\beta F(m)}.$$

where m refers to the average magnetization for a configuration,

$m = -1$ to $+1$ in steps of $\frac{2}{N}$

But in the large N limit

$$Z = \frac{1}{\Delta m} \sum_{m} \Delta m \, e^{-\beta F(m)} = \frac{N}{2} \int_{-1}^{1} dm \; e^{-\beta F(m)}.$$

- We can recover the old F_{thermo} from this
- Note that $F(m)$ is an extensive quantity

  write $F(m) = N f(m)$

$$Z = \int_{-1}^{1} dm \; e^{-\beta N f(m)}$$

In the $N \to \infty$, we can approximate this integral ($N \sim 10^{20}$)

by steepest descent

$Z \approx e^{-\beta N f(m^*)}$ where m^* is the minimum of $f(m)$.

To get a sense of what's going on, we'll make an approximation ...

MEAN FIELD THEORY

[Cardy, Ch. 1]

Let us define $M = \frac{1}{N}\sum s_i$ for a given config.

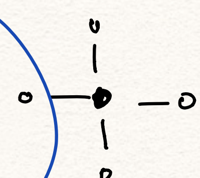
Now let us write $\underset{\pm 1}{s} = M + \overbrace{\delta s}^{\text{not just } \pm 1}$.

$$s_i s_j = (M + \delta s_i)(M + \delta s_j)$$
$$= M^2 + 2M(\delta s_i + \delta s_j) + \delta s_i \delta s_j$$

$$s_i = M + \delta s_i$$

$$H = -J \sum_{ij} s_i s_j - h \sum s_i$$
$$= -\frac{Jq}{2} N M^2 - h N M - \sum J_{ij}\, \delta s_i \delta s_j$$

q : coordination # (# bonds at each site)



Let's ignore this!
We're not sure if it ok yet.
But lets do it anyway.

This mean we are ignoring correlations between fluctuations about the mean.
Clearly, if the correlation length is large then this is not a good approximation, but we will come back to this. Let us just see what happens if we ignore this.

LECTURE #2

Friday, Feb 28, 2020

There were several good points last time.

1) About $f(m)$ being flat

This is actually a very common scenario and forms the basis of the discussion about SSB.

2) About extensivity of the free energy.

This can infact be proved for Ising model (and more general models). The proof is kind of neat and takes inspiration from RG.

3) Clarification on MFT.

Let me clarify the claim of MFT.

$$Z = \sum_{[s]} e^{-\beta E[s]} = \sum_{m} \sum_{[s]|m} e^{-\beta E[s]}$$

Now, $E[s] =$

Given bond $\langle ij \rangle$

$$J s_i s_j = J(m + \delta s_i)(m + \delta s_j)$$

$$= J m^2 + J m(\delta s_i + \delta s_j) + O(\delta s^2)$$

$$= J m^2 + J m(s_i + s_j - 2m) + O(\delta s^2)$$

$$= -J m^2 + J m(s_i + s_j) + O(\delta s^2)$$

$$\sum_{\langle ij \rangle} J s_i s_j = -\frac{J N q}{2} m^2 + J m \, q\left(\sum s_i\right) + O(\delta s^2)$$

$$\left\{ \begin{aligned} &\delta s_i = s_i - m \\ &\delta s_i + \delta s_j = s_i + s_j - 2m. \end{aligned} \right\}$$

In other way of doing this, set $s_i = m$

$$Z(m) = \sum_{[s]|m} e^{+\beta J N q m^2/2 + \beta h N m + O(\delta s^2)}.$$

$$= \sum_{[s]|m} e^{-\beta E(m) + O(\delta s^2)}.$$

$$= e^{-\beta E(m)} \sum_{[s]|m} e^{O(\delta s^2)}$$

$$Z = \frac{\sum_{s} e^{-\beta E(m)} \left(e^{O(\delta s^2)} \right)}{\sum_{[s]} e^{-\beta E(m)}} \sum_{[s]} e^{-\beta E(m)}$$

$$= Z_{MF} \underbrace{\left\langle e^{O(\delta s^2)} \right\rangle_{MF}}_{\text{one can check if this is small! But that will come later.}}$$

OK, now let us see what can MFT tell us.

$$\frac{E}{N} = -Bm - \frac{Jq}{2} m^2$$

Let us compute the effective free energy. $f(m)$.

$$e^{-\beta f(m)} = \sum_{[S]|m} e^{-\beta E(m)}$$

$$= e^{-\beta E(m)} \underbrace{\sum_{[S]|m} 1}$$

$\Omega(m)$: # of configurations with $\frac{\sum s_i}{N} = m$.

$\Omega(m)$ is the entropy factor,

or the # of microstates corresponding to a given macrostate.

We need to compute $\Omega(m)$.

Let $N_\uparrow, N_\downarrow$ be the # of $\uparrow$ and $\downarrow$ down spins

$$N_\uparrow + N_\downarrow = N \quad \text{and} \quad N_\uparrow - N_\downarrow = Nm$$

$$\Rightarrow N_\uparrow = \frac{N}{2}(1+m) \quad \text{and} \quad N_\downarrow = \frac{N}{2}(1-m)$$

$$\underbrace{\Omega(m)}_{=} = {}^N C_{N_\uparrow} = \frac{N!}{\left[\frac{N}{2}(1-m)\right]! \left[\frac{N}{2}(1+m)\right]!}$$

e^{S}

where S is the entropy.

Take the log of both sides and use Sterling's approx

$$N! = N \ln N - N$$

to get

$$S = \ln N! - \ln\left(\frac{N}{2}(1-m)\right)! - \ln\left(\frac{N}{2}(1+m)\right)!$$

$$= N \ln N - \frac{N}{2}(1-m) \ln \frac{N}{2}(1-m) - \frac{N}{2}(1+m) \ln \frac{N}{2}(1+m)$$
$$- \cancel{N} \qquad - \frac{N}{2}\cancel{(1-m)} \qquad - \frac{N}{2}\cancel{(1+m)}$$

$$= \ln \frac{N}{2}\left[\cancel{N} - \frac{N}{2}\cancel{(1-m)} - \frac{N}{2}\cancel{(1+m)}\right] + N \ln 2$$
$$- N\left[(1-m)\ln(1-m) + (1+m)\ln(1+m)\right]$$

$$= N\left[\ln 2 - \frac{1}{2}(1-m)\ln(1-m) - \frac{1}{2}(1+m)\ln(1+m)\right]$$

So the free energy $f(m)$ now becomes

$$f(m) = \frac{1}{N}(E - TS)$$

$$= -Bm - \frac{Jq\,m^2}{2}$$

$$- T \Big[\ln 2 - \underbrace{(1-m)\ln(1-m) - (1+m)\ln(1+m)}_{\text{even in } m.} \Big]$$

We want to look for minima $f(m)$.

We can directly find minima of this expression, or we can restrict ourselves to the case when m is small. $|m| \ll 1$

In that case, we get

$$\frac{(1-m)}{2}\ln(1-m) + \frac{(1+m)}{2}\ln(1+m) = \frac{m^2}{2} + \frac{m^4}{12} + O(m^6)$$

$$\boxed{f(m) = -\underbrace{T\log 2}_{\text{just a constant}} - \underbrace{Bm}_{\text{set to } B=0 \text{ for now}} + \frac{1}{2}\underbrace{(T - Jq)}_{}m^2 + \underbrace{\frac{1}{12}Tm^4}_{\text{Become imp. for large } T.} + \dots}$$

The first term with drama. Sign can flip as we change T.

Recall the question: can this $f(m)$ describe phase transitions?

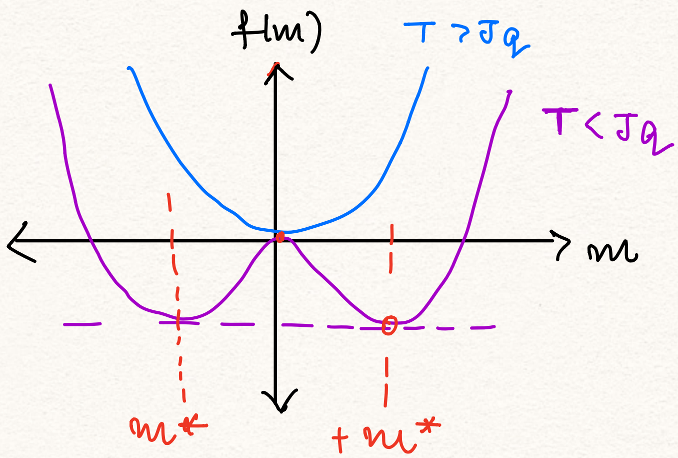
This is <u>Landau approach</u>

Let us first set $\boxed{B=0}$.

Case I $\boxed{B=0 \;:\; \text{Second-order phase transition.}}$

- Let us ignore the additive constant factor. $\log 2$.

$$f(m) = \frac{1}{2}(T - Jq)m^2 + \frac{T}{12}m^4 + \cdots$$



OK, so we see a phase transition!

So, indeed MFT can give us a phase transition.

What about the peculiar power-law behavior of various observables?

1) m as $T \to T_c^{\pm}$?

$f'(m) = 0 \quad \Rightarrow$

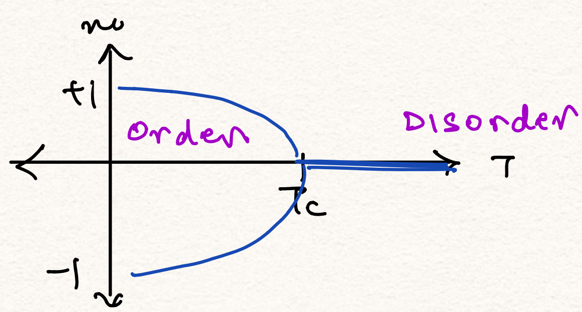
$$m = \left[3\frac{(T - T_c)}{T}\right]^{1/2} \sim t^{1/2}$$

our first critical exponent.

reduced

temperature

We find that



2) Heat Capacity as $T \longrightarrow T_c^{\pm}$

$$C = \frac{\partial \langle E \rangle}{\partial T} = -\frac{\partial}{\partial T}\left(\frac{1}{\beta}\frac{\partial \log Z}{\partial \beta}\right)$$

$$= -\frac{\partial T}{\partial \beta}\,\frac{1}{\partial \beta}\left(\frac{\partial \ln Z}{\partial \beta}\right).$$

$$\boxed{C = +\frac{1}{\beta^2}\frac{\partial^2 \log Z}{\partial \beta^2}}$$

$\log Z$ at $m = m^*$

) when $T > T_c \Rightarrow \log Z = e^{-\beta f(0)} = 1$, with $f(0) \to 0$

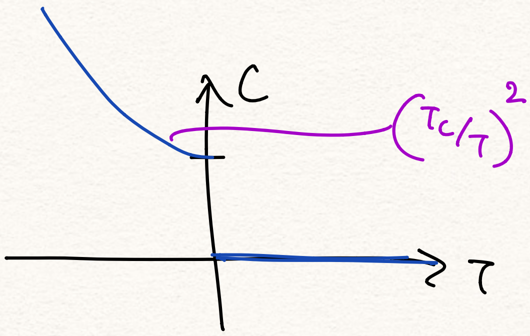
so $C = 0$

2) when $T < T_c \Rightarrow$

$$f(m_0) = -\frac{3}{4}\frac{(T_c - T)^2}{T}$$

$$e^{-\beta f(m_0)} = e^{+\frac{3}{4}\frac{(T_c - T)^2}{T^2}} = e^{3/4(\beta T_c - 1)^2}$$

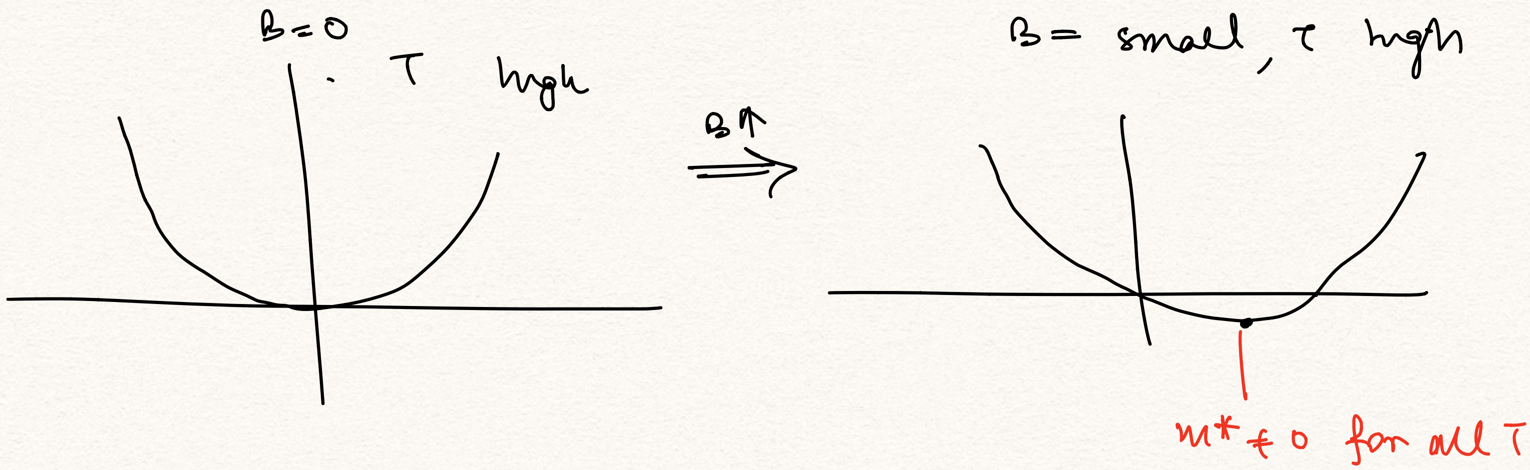
$$C = \frac{3}{2}\left(\frac{\tau_c}{\tau}\right)^{2} \longrightarrow \frac{3}{2} \quad \text{as} \quad \tau \to \tau_c^-$$

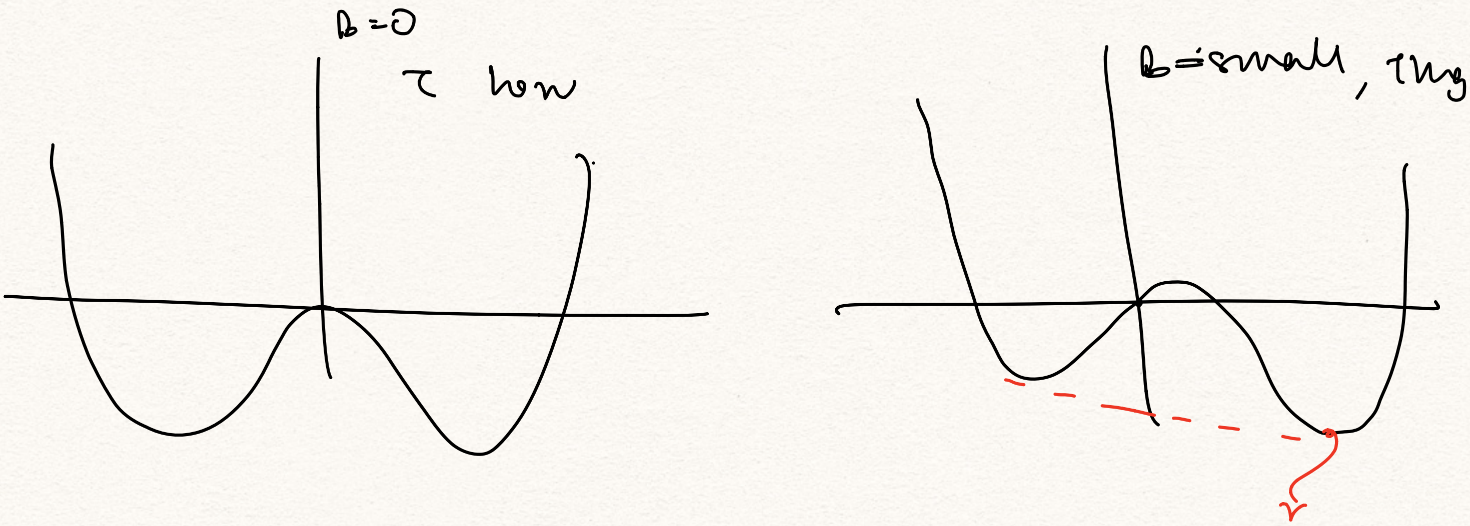


Case II

$$\boxed{B \neq 0 : \text{ First order phase transition}}$$

$$f(m) = -Bm + \frac{1}{2}(T - Jq)m^2 + \cdots$$

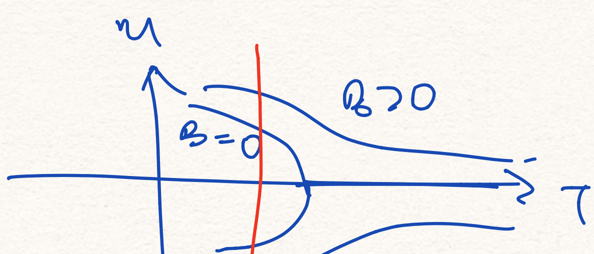




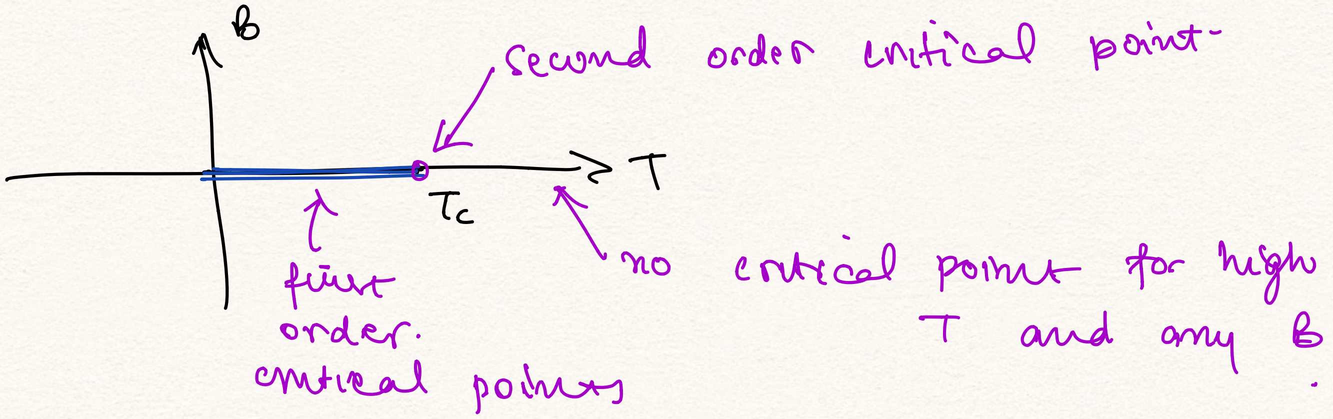
m* picks a value.

No phase transition as we change T.

But below T_c, if we change B from +ve to −ve, the magnetization abruptly goes from +1 to −1.



first order transition



Close to the critical point

1) At $T = T_c$, how does magnetisation change as $B \to 0^+$

$$f(m) = -Bm + \frac{1}{12} T_c m^4$$

$$f'(m) = 0 \Rightarrow -B + \frac{T_c m^3}{3} \Rightarrow m \sim B^{1/3}.$$

2) what about susceptibility

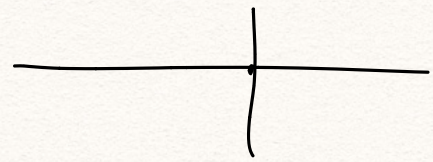
$$\chi = \left.\frac{\partial m}{\partial B}\right|_{B=0} \quad \text{as } T \to T_c^{\pm}$$

a) $T \to T_c^+$, we know $m = 0$

So $$f(m) = -Bm + \frac{1}{2}(T - T_c) m^2.$$

$$\Rightarrow m \approx \frac{B}{T - T_c} \Rightarrow \boxed{\chi = \frac{\partial m}{\partial B} = \frac{1}{T - T_c}}$$

b) $\tau \longrightarrow \tau_c^-$, the minima only come from the m^4 term.



So we keep that

$$f(m) = -Bm + \frac{1}{2}(\tau - \tau_c)m^2 + \frac{1}{12}Tm^4 + \dots$$

$f'(m) = 0 \Rightarrow$ cubic, don't want to solve.

Solve for $B = 0$

$$f'(m) = 0 \Rightarrow -B + tm_* + \frac{Tm_*^3}{3} = 0$$

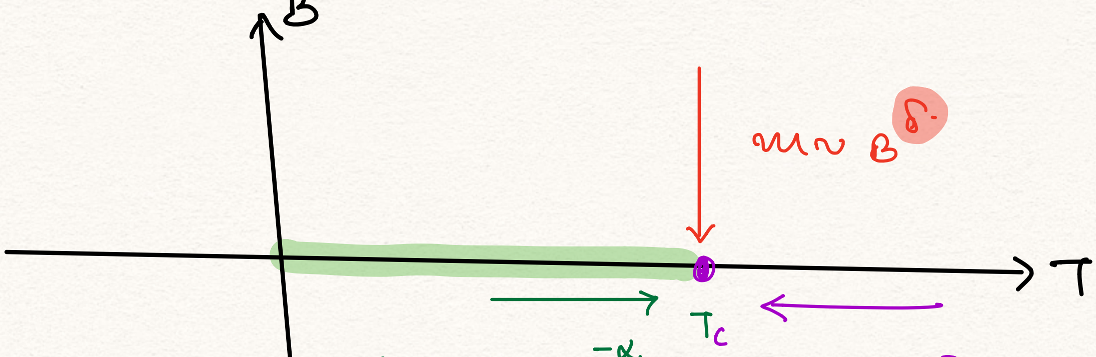
We need to see how $m_*(B)$ changes as $B \uparrow$

$$\frac{\partial}{\partial B} \Rightarrow -1 + \left(3Tm^2 - Tm_0^2\right)\frac{\partial m^*}{\partial B} = 0$$

$$\Rightarrow -1 + 2Tm_0^2 \frac{\partial m^*}{\partial B} = 0$$

$$\chi = \frac{\partial m^*}{\partial B} = \frac{1}{2Tm_0^2} \Rightarrow \boxed{m^* = m_0 + \frac{B}{2Tm_0^2}}$$

THE CRITICAL EXPONENT ZOO.



- $c \sim c_- t^{-\alpha}$
- $m \sim t^{\beta}$
- $\chi \sim \frac{1}{|t|^{\gamma}}$

$c \sim c_+ t^{-}$

$m = 0$

	MF	d=2 (Exact)	d=3.
α	discont.	log. divergence	0.1103
β	1/2	1/8	0.3264
γ	1	7/4	1.2371
δ	3	15	4.7898.

we computed these already

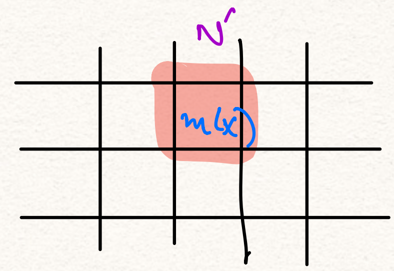
how do we get these ?

Lecture # 3

Landau Ginzburg theory

→ Setting up path integrals.

- Our earlier approach was too crude. It didn't allow for any spatial fluctuations.
- So instead of using a single 'm' for the whole configuration; we can make blocks in a configuration of size N'



So now we write the partition function as

$$Z = \sum_{[s]} e^{-\beta E[s]} = \sum_{m} \sum_{[s]|m} e^{-\beta E[s]}$$

$$= \sum_{\substack{m(x_1) \\ m(x_2) \\ \vdots}} \underbrace{\sum_{[s]|m_1, m_2 \ldots} e^{-\beta E[s]}}_{e^{-\beta F(m_1, m_2, \ldots)}} = \sum_{[m(x)]} e^{-\beta F[m(x)]}$$

We have now a <u>local order parameter</u> $m(x)$.

$$m(x) = \frac{1}{N} \sum_{S_i} S_i'$$

in a block around x

We now use a notation

$$\boxed{Z = \int \mathcal{D}m(x)\; e^{-\beta F[m(x)]}}$$

We have a "path integral" now!

Before we can do anything with this, we need to find what $F[m(x)]$ is.

We can constrain it's form by invoking some general ideas:

1) Locality : $F[m(x)] = \int d^dx\, f(m(x))$
2) Translational and rotational symmetry
3) $\mathbb{Z}_2$ symmetry : $m(x) \rightarrow -m(x)$ for $B=0$
4) Analyticity
5) $m(x)$ is slowly varying (higher derivatives are smaller).

With this, we can now write down the most general form for $f(m(x))$

$$\boxed{f(m(x)) = \alpha_2\, m(x)^2 + \alpha_4\, m(x)^4 + \gamma\, (\vec{\nabla} m(x))^2 + \cdots}$$

The problem is that we don't know what $\alpha_2, \alpha_4, \gamma$ are. We can't compute them from first principles.

We can get some intuition from the expression for Landau energy, where we found

$$\alpha_2 = \frac{1}{2}(T - T_c) \qquad\qquad \alpha_{2_1} = \frac{T}{12}$$

But really all we need to see is that $\alpha_4 > 0$, $r > 0$ and α_2 flips sign at the second order phase transition.

[End of Lecture #3]

Lecture #4

Thursday, March 26, 2020.

- Saddle point approximation $\Rightarrow$ Mean Field Theory.

$$\frac{\delta F}{\delta m(x)} = 0 \Rightarrow -2\gamma \nabla^2 m(x) + 2\alpha_2 m + 4\alpha_4 m^3 = 0$$

The solution with $\nabla m = 0 \Rightarrow m = \text{const}$ correspond to MFT.

Even if there are other solution, they will have $(\nabla m)^2 > 0$ so they will increase $f(m)$

Therefore the lowest minima comes from the $m = \text{const}$ solution, and the saddle point approximation reproduces MFT.

Let's compute the partition function!

$$Z = \int \mathcal{D}\phi(x)\, e^{-\beta F[\phi]}$$

Let us ignore the ϕ^4 term for now.

$$F[\phi(x)] = \frac{1}{2}\int d^dx \left(\gamma \vec{\nabla}\phi\cdot\vec{\nabla}\phi + \mu^2\phi^2\right) \quad \text{——— (i)}$$

Note that our computation should not be trusted near $\mu^2=0$, since the ϕ^4 term becomes important !

To solve (i), we introduce

$$\phi(x) = \frac{1}{V}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}\phi_k$$

in a finite volume.

$$\vec{k} = \frac{2\pi\vec{n}}{L}.$$

We get

$$F[\phi_{\vec{k}}] = \frac{1}{2}\int d^dx \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\left(-\vec{k}_1\cdot\vec{k}_2\gamma + \mu^2\right)\phi_{k_1}\phi_{k_2}\, e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{x}}$$

$$= \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\left(+k^2\gamma + \mu^2\right)\phi_k^*\phi_k.$$

noting $\int d^dx\, e^{i\vec{k}\cdot(\vec{x}_1+\vec{x}_2)} = (2\pi)^d\delta^d(\vec{k}_1+\vec{k}_2)$

and.

$$\int \mathcal{D}\phi(x) = \prod_{k}\int d\phi_k\, d\phi_k^*$$

we get

$$Z = \prod_k \int d\phi_k \, d\phi_k^* \, e^{-\frac{\beta}{2} \frac{1}{V} \sum_k (\gamma k^2 + \mu^2) |\phi_k|^2}$$

$$= \prod_{\substack{k \\ k_1 > 0}} \left[\int d\phi_k \, d\phi_k^* \, e^{-\frac{\beta}{2} (\gamma k^2 + \mu^2) |\phi_k|^2} \right].$$

$$= \prod_{\substack{k \\ k_1 > 0}} \left\{ \int dz \, dz^* \, e^{-\frac{\beta}{2} (\gamma k^2 + \mu^2) |z|^2} \right\}$$

$$= \prod_{\substack{k \\ k_1 > 0}} \frac{2\pi}{\beta(\gamma k^2 + \mu^2)}$$

$$= \prod_k \left[\frac{2\pi T}{\gamma k^2 + \mu^2} \right]^{1/2}.$$

$$\frac{-\beta F_{\text{thermo}}}{V} = \frac{\log Z}{V}$$

$$\Rightarrow \quad f_{th} = -\frac{T}{V} \log Z = -\frac{T}{2V} \sum_k \log\left(\frac{2\pi T}{\gamma k^2 + \mu^2} \right)$$

$$= -\frac{T}{2} \int \frac{d^d k}{(2\pi)^d} \log\left(\frac{2\pi T}{\gamma k^2 + \mu^2} \right)$$

Lecture #5

Friday, March 27, 2020

Correlation functions

$$\langle \phi(x)\,\phi(y)\rangle_c = \langle \phi(x)\,\phi(y)\rangle - \langle \phi(x)\rangle\langle \phi(y)\rangle.$$

Trick:

couple the field ϕ to a "magnetic field" $B(x)$.

$$\Longrightarrow F[\phi] = \int d^d x \left\{ \frac{r}{2}(\nabla\phi)^2 + \frac{\mu^2}{2}\phi^2 + B(x)\,\phi(x) \right\}$$

Show that

1) $$-\frac{1}{\beta}\frac{1}{Z}\frac{\delta Z}{\delta B(x)} = \langle \phi(x)\rangle_B.$$

2) $$\frac{1}{\beta^2}\frac{1}{Z}\frac{\delta^2 Z}{\delta B(x)\,\delta B(y)} = \langle \phi(x)\,\phi(y)\rangle_B.$$

3) $$\frac{1}{\beta^2}\frac{\delta^2 \log Z}{\delta B(x)\,\delta B(y)} = \langle \phi(x)\,\phi(y)\rangle_{B,\,\text{conn.}}$$

Gaussian Integration

1) show that $\int d^N y \, e^{-\frac{1}{2} y^T G^{-1} y} = \det (2\pi G)^{1/2}$.

2) $\int d^N y \, e^{-\frac{1}{2} y G^{-1} y + B^T y} = \det (2\pi G)^{1/2} \, e^{\frac{1}{2} B^T G B}$

3) $\left. \dfrac{\partial^2 \log Z}{\partial B_i \partial B_j} \right|_{B=0} = (G)_{ij}$

In the path integral language, we just import the above results and write

$$\frac{\partial^2 \log Z}{\partial B(x) \partial B(y)} = G(x, y)$$

$$\Rightarrow \boxed{\langle \phi(x) \phi(0) \rangle_c = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i \vec{k} \cdot \vec{x}}}{(\gamma k^2 + \mu^2)}}$$

- We can also think of the 2pt function as the susceptibility.

$$\chi(x, y) = \frac{\partial \langle \phi(x) \rangle_B}{\partial B(y)} = \frac{\partial}{\partial B(y)} \left[\frac{-1}{\beta} \frac{\partial \log Z}{\partial B(x)} \right]$$

$$= -\frac{1}{\beta} \frac{\partial^2 \log Z}{\partial B(y) \partial B(x)} = \frac{-1}{\beta} \langle \phi(x) \phi(y) \rangle_{B, c}$$

$$\boxed{\text{Behavior of the 2-point correlation function}}$$

- $$G(r) = \frac{1}{\gamma} \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2 + 1/\xi^2}$$

where we have defined

a length scale $\boxed{\xi^2 = \frac{\gamma}{\mu^2}}$ Correlation length.

- 2 regimes i) $r \gg \xi$ and ii) $r \ll \xi$

$$G(r) \sim \begin{cases} \dfrac{1}{r^{d-2}} & r \ll \xi \quad \text{Power law decay} \\[2ex] \dfrac{e^{-r/\xi}}{r^{(d-1)/2}} & r \gg \xi \quad \text{Exponential decay} \end{cases}$$

- So correlations die out exponentially for $r \gg \xi_1$;

  therefore ξ_1 determines the characteristic length scale for correlations.

- Recall that $\xi_1 = \frac{\gamma}{\mu^2}$

  and $\mu^2 \sim |T - T_c|$ near the critical point.

  So $\xi_1 \sim \frac{1}{|T - T_c|^{1/2}} \to \infty$ as $T \to T_c$

- Therefore, fluctuations at <u>all</u> <u>length</u> <u>scales</u> become important as we go closer to a 2nd order phase transition!