$m(x) \longrightarrow \phi(x)$ "field"

Lecture #4
March 26, 2020

$$Z = \int \mathcal{D}\phi(x)\, e^{-\beta F[\phi]}$$

$$F[\phi] = \frac{1}{2}\int d^d x \left(\mu_2 \phi^2 + \cancelto{0}{\mu_4 \phi^4} + \cancelto{0}{\cdots} + \gamma(\vec{\nabla}\phi)^2 + \cdots \right)$$

$\downarrow$

- $$F[\phi] = \frac{1}{2}\int d^d x \left(\gamma(\vec{\nabla}\phi)^2 + \mu_2\phi^2 \right)$$

- Free field theory

$$Z = \int \mathcal{D}\phi\, e^{-\beta\int d^d x\, (\gamma(\vec{\nabla}\phi)^2 + \mu_2\phi^2)}$$

$$= \left(\det 2\pi K^{-1}\right)^{1/2} = \prod_i \left(\frac{2\pi}{\lambda_i}\right)^{1/2}$$

$$\int_{-\infty}^{\infty} dy\, e^{-\alpha y^2} = \sqrt{\frac{\pi}{\alpha}}$$

Generalization 1

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\int d^N y \, e^{-\frac{1}{2} y^T A y} = \left\{ \det 2\pi A^{-1} \right\}^{1/2} \quad [N]$$

$N \times N$

Generalization 2.

$N \longrightarrow \infty$ $\qquad$ $y_1, \cdots, y_N \longrightarrow \phi(x_1) \cdots \phi(x_N)$

$$\int \mathcal{D}\phi(x) \, e^{-\frac{1}{2} \int d^d x \, d^d y \, \phi(x) K(x,y) \phi(y)}$$

$$= \left[\det 2\pi K^{-1} \right]^{1/2}$$

λ_i are eigenvalues.

$$\boxed{\det(K) = \prod \lambda_i}$$

$$Z = \int \mathcal{D}\phi \, e^{-\beta \boxed{\int d^d x \left(\gamma (\vec{\nabla}\phi)^2 + \mu_2 \phi^2 \right)}}$$

$$= \left(\det 2\pi K^{-1} \right)^{1/2} = \prod_i \left(\frac{2\pi}{\lambda_i} \right)^{1/2}$$

$$\int d^d x \, \left(\gamma \, \vec{\nabla}_x \phi_x \cdot \vec{\nabla}_x \phi_x + \mu_2 \, \phi^2 \right)$$

$$\int d^d x \, d^d y \; \mu_2 \, \phi(x) \, \delta(x-y) \, \phi(y)$$

$$\int d^d x \, \vec{\nabla}\left(\phi \vec{\nabla}\phi\right) - \phi \, \vec{\nabla}^2 \phi$$

$$\int d^d x$$

$$\int d^d x \; \phi(x) \left(-\gamma \nabla^2 + \mu^2 \right) \phi(x)$$

$$\int d^d x \, d^d y \; \phi(x) \underbrace{\left(-\gamma \nabla_x^2 \delta(x-y) + \mu^2 \delta(x-y) \right)}_{K(x,y)} \phi(y)$$

$$\int d^d y \; K(x,y) \, \psi(y) = \lambda \, \psi(x)$$

eigenfunction eigenvalue

$$\left(-\gamma \nabla^2 + \mu^2 \right) \psi(x) = \lambda \, \psi(x)$$

$$\psi_{\vec{k}}(x) = e^{i\vec{k}\cdot\vec{x}}$$

$$\beta(-\gamma\nabla^2 + \mu^2)\,\psi_{\vec{k}}(x)$$

$$= (-\gamma(i\vec{k})^2 + \mu^2)\,\psi_{\vec{k}}(x)$$

$$= \underbrace{\beta(\gamma\vec{k}^2 + \mu^2)}_{\lambda_{\vec{k}}}\,\psi_{\vec{k}}(x)$$

$$\psi_{\vec{k}}(x) = e^{i\vec{k}\cdot\vec{x}} \longleftrightarrow \lambda_{\vec{k}} = \beta(\gamma\vec{k}^2 + \mu^2)$$

$$\vec{k} = \frac{2\pi}{L}\vec{n} \qquad \vec{n} = (n_1, \dots, n_d)$$

$$\vec{n} \in \mathbb{Z}^d$$

$$\boxed{f(x) = \int d^d k \; \tilde{f}(k)\, e^{ikx}} \qquad e^{i\vec{k}\cdot\vec{x}}$$

$$\int dx \; |f(x)|^2 \longrightarrow \text{finite}$$

$$Z = \prod_{\lambda} \left(\frac{2\pi}{\lambda}\right)^{1/2}$$

$$\lambda_{\vec{k}} = \gamma k^2 + \mu^2.$$

$$= \prod_{\vec{k}} \left(\frac{2\pi}{\beta(\gamma k^2 + \mu^2)}\right)^{1/2}$$

$$Z = e^{-\beta F_{thermo}}$$

$$-\beta F_{th} = \log Z = \sum_{\vec{k}} \frac{1}{2} \log \frac{2\pi}{\beta(\gamma k^2 + \mu^2)}.$$

$$F_{th} = V f_{thermo}$$

$$f_{th} = \frac{1}{-2\beta} \frac{1}{V} \sum_{\vec{k}} \log \frac{2\pi}{\beta(\gamma k^2 + \mu^2)}$$

$$\downarrow$$

$$\int \frac{d^d k}{(2\pi)^d}.$$

$$\boxed{f_{th} = -\frac{1}{2\beta} \int \frac{d^d k}{(2\pi)^d} \log \frac{2\pi}{\beta(\gamma k^2 + \mu^2)}}$$

$$e^{-\beta F[\phi]} \longleftrightarrow e^{iS[\phi]}$$

1) Heat Capacity per unit volume

2) Correlation $\langle \phi(x) \; \phi(y) \rangle = ?$

$\uparrow$ $\langle \phi(x) \phi(x+a) \rangle =$

<u>Heat capacity</u>

$\boxed{HW}$

$$c = \frac{C}{V} = \frac{1}{V}\frac{\partial \langle E \rangle}{\partial T} = \beta^2 \frac{\partial}{\partial \beta^2} \log Z$$

$$= \beta^2 \frac{\partial}{\partial \beta^2}\left\{ -\frac{\beta V}{2} \int \frac{d^d k}{(2\pi)^d} \log \frac{2\pi}{\beta(r k^2 + \mu^2)} \right\}$$

$\mu^2 \sim T - T_c$ $\qquad r = \text{const.}$

$$c = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \left\{ \cancel{1} - \frac{2\pi T}{r k^2 + \mu^2} + \frac{T^2}{(r k^2 + \mu^2)^2} \right\}$$

(1) $\qquad$ (2)

$$\mu^2 = T - T_c$$

$$(1) = 2\pi T \int \frac{d^d k}{(2\pi)^d} \frac{1}{\gamma k^2 + \mu^2} \; — \begin{cases} 1/\mu & d = 1 \\ \Lambda^{d-2} & d \geq 2 \end{cases}$$

$d = 1$ $\qquad \int_0 \frac{dk}{\gamma k^2 + \mu^2} \sim \frac{1}{\mu}$ $\qquad$ finite

$d = 2$ $\qquad ? \quad \int d^2k / \gamma k^2 + \mu^2 \sim \log(\;)$

$d = 3$ $\qquad \int \frac{d^3 k}{\gamma k^2 + \mu^2} \sim \int \frac{d^3 k}{k^2} \sim \Lambda$

d $\qquad \Lambda^{d-2}$

$$(2) = T^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(\gamma k^2 + \mu^2)^2} \sim \begin{cases} \mu^{d-4} & d < 4 \\ \Lambda^{d-4} & d \geq 4 \end{cases}$$

$$\sim 1/k^4$$

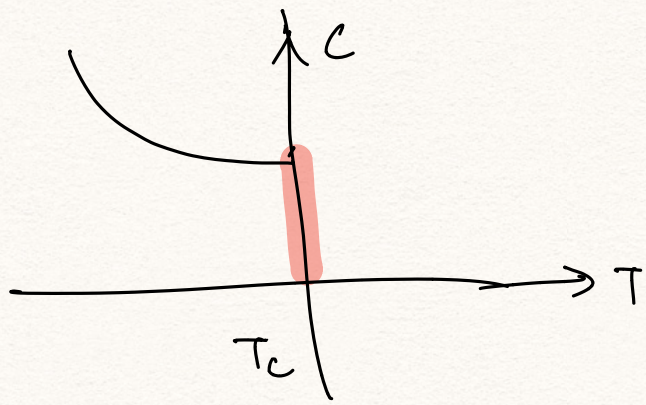
$d = 3$ $\qquad \frac{1}{\mu}$

$\underline{d < 4}$ $\qquad \mu \to 0$

$$C \sim (\mu^2)^{\frac{d-4}{2}} \sim (T - T_c)^{\frac{d-4}{2}}$$

$\sim \quad |T - T_c|^{-\alpha}$

$$\boxed{\alpha = \frac{4-d}{2}}$$



Lecture # 5
March 27, 2020

$$Z = \int \mathcal{D}\phi \, e^{-\beta F[\phi]}$$

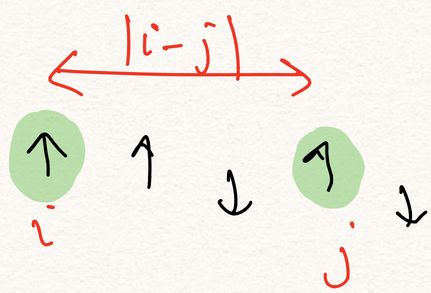
$$F[\phi] = \frac{1}{2} \int d^d x \left(\gamma (\vec{\nabla}\phi)^2 + \mu^2 \phi^2 \right).$$

$$= \frac{1}{2} \int d^d x \, d^d y \quad \phi(x) \, K(x, y) \, \phi(y)$$

$$Z = \det \left(2\pi K^{-1}\right)^{1/2}$$

Correlation functions

$$\langle \phi(\vec{x}) \, \phi(\vec{y}) \rangle = \langle \phi(|\vec{x} - \vec{y}|) \, \phi(0) \rangle$$



$$\langle \phi(\vec{r}) \, \phi(0) \rangle = G(|\vec{r}|) = G(r)$$

Trick.

$$Z[B(x)] = \int \mathcal{D}\phi \, e^{-S[\phi] + \int d^d x \, B(x) \phi(x)}$$

$$Z = \int d^N y \; e^{S[y] + \sum_i B_i y_i}$$

$$\frac{\partial Z}{\partial B_i} = \int d^N y \quad y_i \quad e^{S[y] + \sum_i B_i y_i}$$

$$\frac{1}{Z} \frac{\partial^2 Z}{\partial B_i \partial B_j} = \frac{\int d^N y \, (y_i y_j) \, e^{S[y] + \sum B_i y_i}}{\int d^N y \quad e^{S[y] + \sum B_i y_i}}$$

$$= \langle y_i y_j \rangle_B$$

$$\boxed{\langle y_i y_j \rangle = \frac{1}{Z} \frac{\partial^2 Z}{\partial B_i \partial B_j} \Bigg|_{B=0}}$$

$$\frac{\partial^2 \log Z}{\partial B_i \partial B_j} = \langle y_i y_j \rangle - \langle y_i \rangle \langle y_j \rangle$$

$$= \langle y_i y_j \rangle_{\text{connected}}$$

$$Z = \int d^N y \quad e^{-\frac{1}{2} y^T K y} = \det(2\pi K^{-1})^{1/2}$$

$$\Big\downarrow \text{H.W.}$$

$$\int d^N y \, e^{-\frac{1}{2} y^T K y + B^T y}$$

$$Z = \det (2\pi K^{-1})^{1/2} \, e^{\frac{1}{2} B^T K^{-1} B}.$$

$$Z[B] = \int \mathcal{D}\phi \; e^{-\beta F[\phi] - \beta \int d^d x \, B(x) \, \phi(x)}$$

$$\frac{\delta^2 \log Z}{\delta B(x) \, \delta B(y)} = \langle \phi(x) \, \phi(y) \rangle_{\text{conn.}}$$

$$Z = \det (2\pi K^{-1})^{1/2} \, e^{\frac{1}{2} B^T K^{-1} B}.$$

$$\log Z = \underbrace{\frac{1}{2} B_i (K^{-1})_{ij} B_j}_{} + \underbrace{\log \left[\det (\;) \right]}_{\text{indep of } B}$$

$$\frac{\partial^2 \log Z}{\quad} = (K^{-1})_{ij} = \langle y_i \, y_j \rangle$$

$\partial B_i \; \partial B_j$

$$K(x,y) = -\nabla_x^2\left(\delta(x-y)\right) + \mu^2$$

$$K^{-1}(x,y) = G(x,y)$$

$$\langle \phi(x)\, \phi(y) \rangle = G(x,y)$$

$$\int d^d y \; K(x,y) \boxed{G(y,z)} = \delta(x-z)$$

$$K_{ij}\, G_{jk} = \mathbb{1}_{ik}$$

Green's function

$$\Rightarrow \int d^d y \; K(x,y)\, G(y,0) = \delta(x)$$

$$\Rightarrow \left(-\gamma \nabla_x^2 + \mu^2\right) G(x) = \delta(x)$$

Fourier transform

$$\left(+\gamma k^2 + \mu^2\right) \tilde{G}(k) = 1$$

$$\Rightarrow \tilde{G}(k) = \frac{1}{\quad}$$

$\gamma k^2 + \mu^2$

$$\boxed{G(x) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{x}}}{\gamma k^2 + \mu^2}}$$

$$\langle \phi(x)\,\phi(y) \rangle_{conn} = \int \frac{d^d x}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{x}}}{\gamma k^2 + \mu^2}$$

$$G(x) = \frac{1}{\gamma} \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{x}}}{k^2 + \frac{\mu^2}{\gamma}}$$

$$= \frac{1}{\gamma} \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{x}}}{k^2 + 1/\xi^2}.$$

$$\xi^2 = \frac{\gamma}{\mu^2}$$ length scale

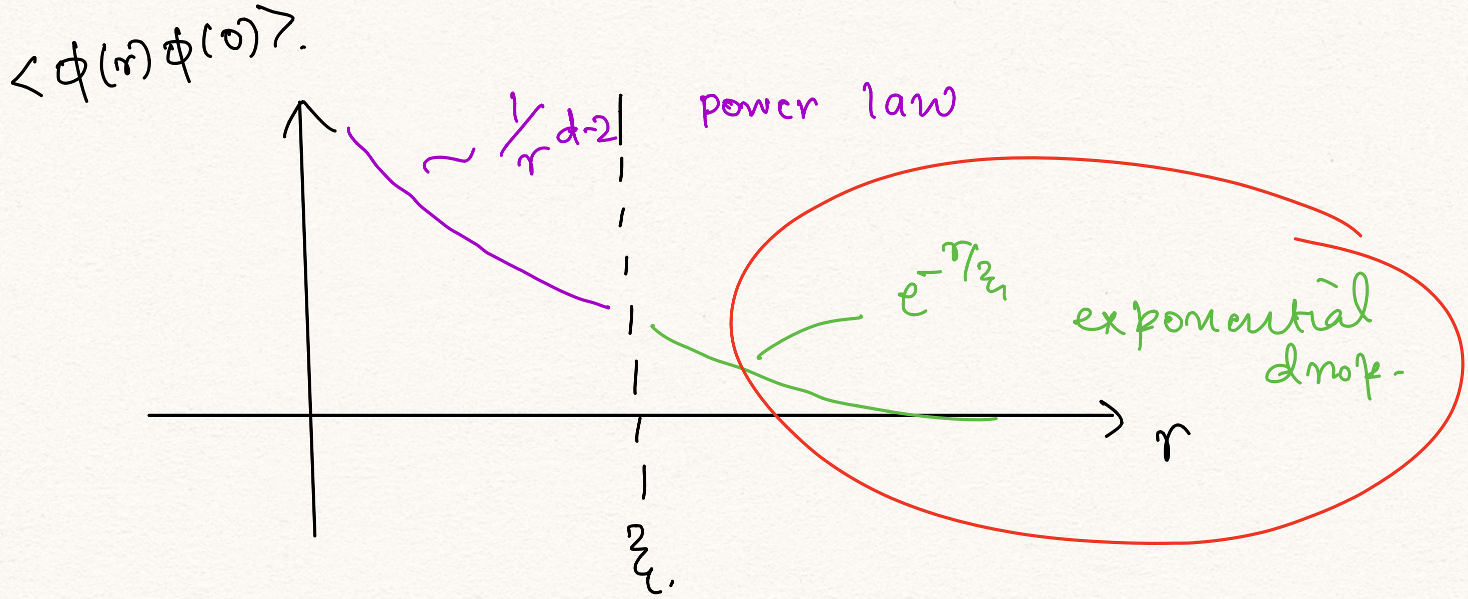
Correlation length

$G(x)$ in 2 regimes

i) $x \ll \xi$

ii) $x \gg \xi$.

$$G(r) \sim \begin{cases} \dfrac{1}{x^{(d-2)}} & x \ll \xi \\[2em] \dfrac{e^{-r/\xi}}{r^{(d-1)/2}} & x \gg \xi \end{cases}$$



$$\xi^2 = \frac{r}{\mu^2} \qquad \mu^2 \sim |T - T_c|$$

$$\xi \sim \frac{1}{|T - T_c|^{1/2}} \longrightarrow \infty \quad \text{when } T \to T_c$$

Fluctuations on all length scales are important as $T \to T_c$!!

Breakdown of Mean Field Theory

$$H = \sum_{ij} s_i s_j \qquad s_i = m_0 + \delta s_i$$

$$\sim \sum m_0^2 + \sum_{ij} \delta s_i \, \delta s_j$$

$$\frac{\left\langle \sum_{ij} \delta s_i \, \delta s_j \right\rangle}{\left\langle \sum_{ij} m_0^2 \right\rangle} \ll 1$$

$$R = \frac{\int_0^{\xi} d^d x \, \langle \phi(x) \phi(0) \rangle}{\int_0^{\xi} d^d x \, m_0^2}$$

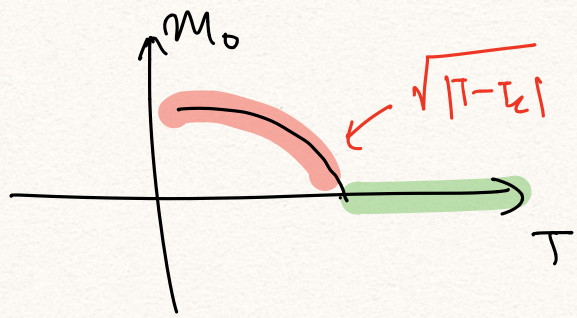
$$\frac{1}{x^{d-2}}$$

$$\sim \frac{\int_0^{\xi} dx \cdot x^{-1} \quad x^2}{m_0^2 \int_0^{\xi} d^d x} \sim \frac{\int dx \; x}{m_0^2 \int d^d x}$$

$$\sim \frac{\xi^2}{m_0^2 \quad \xi^d} \sim \frac{\xi^{2-d}}{m_0^2}$$

$$\xi^2 \sim \frac{r}{\mu^2} \Rightarrow \xi \sim \frac{1}{|T-T_c|^{1/2}} \sim t^{-1/2}$$

$$m_0 \sim t^{1/2}$$



$$R = t^{-\frac{1}{2}(2-d)-1}$$

$$= t^{d/2 - 2} = t^{\overset{z}{\frac{d-4}{2}}}$$

1) $d > 4 \Rightarrow z > 0 \Rightarrow R \to 0$ as $t \to 0$

2) $\boxed{d < 4 \quad \Rightarrow \quad Z < 0 \quad \Rightarrow \quad R \longrightarrow \infty \quad \text{as} \quad z \to 0}$

3) $d = 4 \quad \Rightarrow \quad ??$