AC circuits

Capacitor $Q = CV$, $C = \frac{\epsilon_0 \epsilon A}{d}$

$[F] = \left[\frac{\text{Coulomb}}{\text{Volt}}\right]$  

Practical units: $\mu$F, pF

Electrolytic capacitors: $\mu$F scale, polar $\parallel$, lossy, not good for signals

NP&Ø small drift.

RC circuit

$V = IR$, $I = -\frac{dQ}{dt}$  

$Q/C = -R\frac{dQ}{dt}$  

$Q = -V/R C$

$Q, V, I = Q_0, V_0, I_0 e^{-t/\tau}$, $\tau = RC$

The larger $C$ (more charge stored) or $R$ (the slower discharge) the longer $I$

Capacitor prevents spikes in $V$:  

$V = \frac{\dot{Q}}{C} = \frac{I}{C}$, so spike in $V$ would require $I \to \infty$
Inductors

\[ V = L \frac{dI}{dt} \] back electromotive force

trying to prevent changes in \( I \)

Thus its action is complementary to

that of \( C \).

Units: \[ [H] = \text{Volt/Amp/Sec} \] typically

\( \mu \text{H} \) to \( m\text{H} \)

\[ \begin{align*}
Q(t) & = I(t) + \frac{1}{R} \int I(t) dt \\
\frac{dQ}{dt} & = L \frac{dI}{dt} + R \frac{dQ}{dt} + \frac{Q}{C} = 0 \\
m \ddot{x} + \delta \dot{x} + k x &= 0
\end{align*} \]

- just like damped oscillator

\[ \begin{align*}
x' & = Q \\
v & = \dot{Q} = -I \\
\alpha & = \frac{Q}{I} \\
& \text{inertia; large} \ L \text{ prevents changes in velocity} \\
& \text{damping constant} \\
k & = \frac{1}{C} \text{ spring constant} \\
& \text{under damped: } \ R^2 < 4L/C \\
Q & = Q_0 e^{-\delta t} \cos(\omega_0 t + \phi), \text{ where} \\
\omega_0 & = \sqrt{\frac{1}{C} - \frac{R^2}{4L}} \text{, } \delta = \frac{2L}{R} \text{ and} \\
n & \text{are determined from boundary conditions} \]
AC signals

So far, we considered homogeneous solutions—w/o driving force. Now we add a source of AC voltage $V = V_0 \cos \omega t$

Euler formula: $e^{i\theta} = \cos \theta + i \sin \theta$

$\begin{align*}
\text{Im} & \quad e^{i\theta} \\
\text{Re} & \quad \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \\
\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}
\end{align*}$

Due to the circuit linearity, we can consider the effect of $e^{i\omega t}$ and $e^{-i\omega t}$ separately and contributions. Alternatively, we can some we consider $e^{i\omega t} = \cos \omega t + i \sin \omega t$ keeping the real part only at the end.

Impedance: take $V = V_0 e^{i\omega t}$

$Q = CV \Rightarrow I = C \frac{d}{dt} V_0 e^{i\omega t}$

$Z_C = \frac{V}{I} = \frac{V_0 e^{i\omega t}}{i\omega CV_0 e^{i\omega t}} = \frac{1}{i\omega C}$

Similarly for inductors, take $I = I_0 e^{i\omega t}$

$V = L \frac{d}{dt} I \Rightarrow V = i\omega LI_0 e^{i\omega t} \quad Z_L = i\omega L$
Impedances replace resistances in AC circuits. Impedances add just like resistances. Consider, e.g.,

\[ \frac{1}{C_1 + C_2} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{\text{i} \omega C_1} + \frac{1}{\text{i} \omega C_2} = \frac{1}{\text{i} \omega (C_1 + C_2)} \]

Since \( \frac{1}{\text{i} \omega C_{\text{tot}}} \Rightarrow C_{\text{tot}} = C_1 + C_2 \), as it should be since \( C_1 || C_2 \) hold 2x more charge than single \( C \) (assuming \( C_1 = C_2 \)).

Example: low pass filter

\[ V_{\text{in}} \quad \overset{I}{\longrightarrow} \quad V_{\text{out}} \quad I = \frac{V_{\text{in}}}{R + \frac{1}{C}} \quad V_{\text{out}} = \frac{Z_c}{Z_c + R} V_{\text{in}} \quad \text{and} \quad I = \text{i} \omega C \]

By the way, \( H = \frac{V_{\text{out}}}{V_{\text{in}}} \) is called "transfor function" (more later)

\[ H = \frac{1}{\text{i} \omega C + R} = \left( 1 + \text{i} \omega CR \right)^{-1} \]

\( H \approx 1 \) up to \( w \sim 1 / CR \), then falls off \( \propto 1 / w \) and a phase shift develops:

\[ V_{\text{out}} = H V_{\text{in}} \approx V_{\text{in}} / \text{i} \omega CR = \frac{1}{\text{i} \omega CR} V_{\text{in}} \text{e}^{\text{i}(\omega t - \pi/2)} \]

where \( \text{e}^{\text{i} \pi/2} = \text{i} \). Take Re part: \( \cos(\omega t - \pi/2) \).
Phase

\[ V(\omega, t) = Z(\omega) I(\omega, t) \quad \text{+ dependence: } e^{i\omega t} \]

\[ |V| = |Z| |I| \]

\[ Z = |Z| e^{i\phi} \]

For \( Z_R = R \), \( \phi = 0 \): \( I \) and \( V \) are in phase

\[ Z_L = e^{i\pi/2} \omega L, \quad \phi = \pi/2 \]

Voltage leads current, current lags,

\[ Z_C = e^{-i\pi/2} / \omega C, \quad \phi = -\pi/2 \]

Voltage lags with respect to current

\[ Z = R + iX \quad \text{(resistance + reactance)} \]

Resonant circuit:

\[ Z_{tot} = R + i\omega L - i/\omega C \]

One can add impedances like phasors.

\[ V_{out} = V_{in} R_{tot} \]

\[ V_{out} = \frac{R \cdot V_{in}}{R + ic\omega L - i/\omega C} \]

On resonance, \( cL = 1/\omega C \), the two impedances cancel and \( H = 1 \)
At either $\omega \to 0$ or $\omega \to \infty$, $H \to 0$

as either $C$ or $L$ block the current:

$H \to iwRC$ or $H \to R/\omega L$

$\theta \xrightarrow{\omega \to \infty} 90^\circ$ Notice that the phase goes
from $+90^\circ$ (C-R) through 0 on resonance to $-90^\circ$ (L-R)

Other LCR circuits:

\[
Z_{LIC} = \frac{1}{i\omega C - \frac{1}{i\omega L}} \quad \xrightarrow{\omega \to 0 \text{ or } \infty} \infty \text{ on resonance}
\]

\[
H = \frac{Z_{LIC}}{R + Z_{LIC}} \xrightarrow{\omega \to 0 \text{ or } \infty} 1 \text{ resonance}
\]

\[
H = \frac{Z_L + Z_C}{R + Z_L + Z_C} \xrightarrow{\omega \to 0 \text{ or } \infty} 0 \text{ resonance}
\]

"Notch filer"

\[
H \xrightarrow{\omega_p} 1
\]
Quality factor

\[ Q = \frac{X_L(\omega_0)}{R} = \frac{1}{R} \sqrt{\frac{L}{C}} = \frac{1}{\sqrt{LC/R}} \]

which is also equal to \( \frac{X_L(\omega_0)}{R} = \frac{1}{\sqrt{LC/R}} \)

Physical significance:

in a homogeneous solution (\( \text{Vext} = 0 \))

\[ V_0 e^{-\beta t} \cos \omega_0 t \text{ with } \omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \]

if \( R^2/2L \ll \frac{1}{LC} \iff \frac{L}{CR^2} = \frac{Q^2}{2} \gg 1 \) and \( \omega_0 \approx \omega_r \)

then also \( 6 = R/2L \ll \omega_0 \approx \sqrt{1/LC} \Rightarrow Q \gg 1 \)

Consider \( V \) at the maximum of consecutive oscillations,

\[ \text{Max } V \Rightarrow \text{max } Q \Rightarrow I = \dot{Q} = 0 \text{ and all the energy is stored in the capacitor: } CV^2/2 \]

Consider energy loss per period/over energy:

\[ \frac{\left[ V_0 e^{-2\beta nT} - V_0 e^{-2\beta (n+1)T} \right]}{V_0 e^{-2\beta nT}} = 1 - e^{-2\beta T} = 2\beta T = 2\beta \frac{2\pi}{\omega_0} = 2\pi \frac{R}{L} \sqrt{LC} = 2\pi/\omega_0 \]
Rewrite $H$ of the RCL circuit using $\omega_r$ and $Q$ (only 2 constants are needed, as $H$ depends only on variables $L/R$ and $1/\omega$)

$$H = \frac{R}{R + j\omega L - j\omega C} = \frac{1}{1 + jQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)} \quad (*)$$

Bode plot

- Low $\omega$: $H \approx \frac{i\omega}{Q\omega_r}$
- High $\omega$: $H \approx \frac{\omega_r}{iQ\omega}$
- Slope: $\pm 6\text{ dB/octave}$

"Octave" factor of 2 in $\omega$. $\log_{10} 2 \approx 0.3$

$\text{dB} = 20 \log_{10} H \quad (10 \log_{10} \text{Power}, \text{Pa} \text{V}^2)$

So per octave, $H$ changes by 2. $20\log_{10} 2 = 6$

In general, one can present $H = \frac{P_n(i\omega)}{D_m(i\omega)}$ (ratio of polynomials). At $\omega \to 0$ or $\omega \to \infty$ the last or the first terms are leading, and $H \propto (i\omega)$ some power. Other terms may be leading at intermediate frequencies. (Like $1\text{ in } \times$ for $Q \ll 1$, see figure.)
Power in AC circuits

\[ P(t) = I(t)V(t) \] that includes energy flow
to/from reactive elements (C & L) and one
is typically interested in power averaged a cycle:

Let \( V = V_0 \cos(\omega t + \phi) \), \( I = I_0 \cos \omega t \)

\[
\langle P \rangle = \frac{1}{T} \int_0^T I_0 V_0 \cos(\omega t + \phi) \cos(\omega t) \, dt =
\]
\[
= \frac{I_0 V_0}{2} \left( \cos^2 \omega t \cos \phi - \sin \omega t \cos \omega t \sin \phi \right) \, dt
\]
\[
= \frac{1}{2} I_0 V_0 \cos \phi = I_{\text{rms}} V_{\text{rms}} \cos \phi
\]

\[ V_{\text{rms}} = \frac{V_0}{\sqrt{2}} = \frac{V_{p-p}}{2\sqrt{2}} \]

Fourier transform

Any periodic function \( f(t+T) = f(t) \)

\[
f(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos(n\omega t) + b_n \sin(n\omega t) \right]
\]

Take \( A = 2 \) in Fourier shift
\( \text{and phase} \).

\[ \text{Examples: for } f(t) = \cos \omega t \Rightarrow a_1 = \frac{1}{T_2} \int_{-T_2}^{T_2} \cos \omega t \, dt = \frac{T_2}{2} \]

In figure: all \( a_n = 0 \) \( \text{ (even x odd } dt = 0 \) \text{ for odd}

\[ b_n = \frac{2}{T_2} \int_{-T_2}^{T_2} \sin(n\omega t) \, dt = \frac{1}{n} \int_{0}^{\pi} \sin nx \, dx = \frac{-1}{n} \cos nx \big|_{0}^{\pi} = \frac{2}{n} \]
**Filters**

$$H = \frac{V_{in} \omega C}{1 + i\omega CR} = \frac{1}{i\omega CR + 1}$$

$$H(\omega \gg \frac{1}{RC}) \approx -i\omega \cdot \frac{1}{RC} = e^{-\frac{i\pi}{2}}$$

Phase shift $-90^\circ$; slope $-6\text{dB/octave}$

At corner frequency $|H| = \sqrt{HH^*} = \sqrt{\frac{1}{1 + (i\omega CR)^2}} = \frac{1}{\sqrt{2}} (-3\text{dB})$

$$H_{\text{HP}} = \frac{R}{R + V_{in} \omega C} = \frac{i\omega CR}{1 + i\omega CR}$$

Phase shift $90^\circ$, slope $+6\text{dB/octave}$

**Two stage filters**

$$V_1 = V_{in} \frac{Z_c || (Z_c + R)}{Z_c || (Z_c + R) + R}, \quad V_{out} = \frac{Z_c}{Z_c + R} V_1$$

$$H_{2\text{stage}} \frac{V_{out}}{V_{in}} = \frac{Z_c}{Z_c + R} \frac{Z_c || (Z_c + R)}{Z_c || (Z_c + R) + R}$$

$$= \frac{1}{1 + i\omega CR} \left(1 + R \frac{i\omega C + \frac{1}{R} + \frac{1}{i\omega C}}{1 + R \frac{i\omega C + \frac{1}{R} + \frac{1}{i\omega C}}}ight) = \frac{1}{(i\omega CR)^2 + 3i\omega CR + 1}$$

12dB/octave - Better than 1 stage

Notice: $H_{2\text{stage}}^{\text{LP}} \neq (H_{1\text{st}}^{\text{LP}})^2 = (i\omega CR)^2 + 2i\omega CR + 1$ (extra suppression at $\omega = 1/CR$, often unwanted)
\( R \implies \frac{1}{ioC} \) compared to previous.

\[
H_{\text{stage}}^{\text{HP}} = \frac{1}{(ioCR)^2 + \frac{1}{ioCR} + 1} = \frac{1}{(ioCR)^2 + 3ioCR + 1}
\]

Again, \( H_{\text{stage}}^{\text{HP}} \neq (H_{\text{stage}}^{\text{HP}})^2 \) because 2nd stage loads the first. To reduce this effect, take

\( R \rightarrow 100R \), \( \omega_c \rightarrow 100 \omega_c \) \( (C \rightarrow C/100) \) in the 2nd stage.

\( H \) of the 2nd stage won't change, and the load on the 1st stage will be negligible.

Band pass

\[ H = \frac{ix RC\omega}{x(xRC\omega)^2 + i(x+2)RC\omega + 1} \]

for \( x >> 1 \), \( \omega\ll\frac{1}{RC} \), \( \frac{1}{2RC} < \omega < \frac{1}{RC} \).

\( \frac{1}{2RC} \ll \omega \ll RC \)

\( \log H \)

\[ \frac{1}{2RC} < \omega < \frac{1}{RC} \]

This term enables band pass

with \( H \approx \frac{2x}{x+2} \approx 1 \). For \( x=1 \) this regime is absent, and for \( \omega = \frac{1}{RC} \) \( H = \frac{1}{1 + 3i + 1} = \frac{1}{3} \uparrow \log H \)

Why does it work with \( x >> 1 \) ? By 2nd stage

>> 2nd stage and does not drain much current from it.
Complex frequency, poles & zeroes
Consider complex frequency \( s = \sigma + j\omega \).
Replace all \( j\omega \) in \( H \) by \( s \) to obtain \( H(s) \).
It coincides with \( H(\omega) \) for \( \sigma = 0 \).
For R, C, L circuits, \( H \) is a ratio of polynomials
\[
H(s) = \frac{P(s)}{D(s)} = \frac{A}{n} \frac{(s-a_1)(s-a_2)\ldots(s-a_n)}{(s-b_1)(s-b_2)\ldots(s-b_m)}
\]
\( a_i \) - zeroes, \( b_i \) - poles

\( j\omega \) Frequency response is evaluated
along \( s = j\omega \) axis, and nearby
\( a_i \) 's & \( b_i \) 's have the largest effect.
\[
|H(s)| = |A| \frac{\prod_{i=1}^{n} |s-a_i|}{\prod_{i=1}^{m} |s-b_i|}
\]
Each \( |s-a_i| \) or \( |s-b_i| \) is a distance from \( s \)
to \( a_i \) or \( b_i \).
For example, for high pass filter \( H = \frac{s}{s + 1/RC} \)
\( \omega \) Zero at \( s = 0 \) & pole at \( s = -1/RC \).
For \( \omega < 1/RC \) zero dominates,
For \( \omega >> 1/RC \) their effects (almost) cancel, \( H \to \frac{s}{s + 1/RC} \).
Example: RCL Bandpass

\[ H = \frac{R}{R + sL + \frac{1}{sC}} = \frac{RCS}{s^2LC + RCS + 1} \]

If overdamped, \( Q << 1 \iff RC \gg \sqrt{LC} \)

\[ \begin{array}{c}
\text{log } H \\
\text{vs } R/L \log w
\end{array} \]

If underdamped, \( Q >> 1 \)

\[ \omega_0 = \frac{1}{\sqrt{LC}} \]

Rewrite in terms of \( \omega_r \) & \( Q = \sqrt{\frac{L}{C}} / R \)

\[ H = \frac{1}{1 + iQ \left( \frac{w - \omega_r}{\omega_r} \right)} = \frac{s\omega_r Q}{s^2 + s\omega_r Q + \omega_r^2} \]

zero at \( s = 0 \) and poles at

\[ s/\omega_r = -2Q \pm \sqrt{4Q^2 - 1} \]

\( Q_c = 1/2 \) (or \( R^2 = 4LC \)) - critically damped,

when 2 poles come together. For \( Q >> 1 \) there are

2 poles at \( s = \pm i\omega_r - \omega_r/2Q \) (resonance at \( w = \omega_r \))

For \( Q << 1 \), one pole approaches \( \omega = 0 \), thus

cancelling zero, another pole goes to \( -\infty \).
Integrator & Differentiator

For an ideal differentiator, $V_{out} \propto \frac{dV_{in}}{dt} \propto j\omega V_{in}(t)$ (if $V_{in}(t) \propto e^{j\omega t}$). Therefore $H_{\text{diff}} \propto j\omega$.

Similarly, for an ideal integrator, $H_{\text{int}} \propto \frac{1}{j\omega}$

$V_{out} \propto \int_{-\infty}^{t} V_{in}(t) \, dt = V_{in}/j\omega + \text{const (neglected)}$

These behaviors can be approximated by RC filters.

$\log H$

ideal int.
real int.

ideal diff.
real diff.

Fourier spectrum of $V_{eq}$

$\omega_0 \leq \omega \leq \omega_0$

Take a square wave $V_{sq}(t) = \sum_{n=1, \text{odd}}^{\infty} \frac{A}{n} \sin(n\omega_0 t)$

If $\omega_0 \gg 1/RC$ and all $n\omega_0 \gg 1/RC$, RC filter behaves like an integrator producing triangular wave, which happens to be $\propto \frac{B}{n^2} \cos(n\omega_0 t)$ (\propto \text{integral of } V_{eq}(t)$).

For $\omega_0 \leq 1/RC$, there are significant distortions, as RC no longer works as an ideal integrator.
Differentiator is more tricky.

\[ V_{\text{diff}} = \frac{d}{dt} V_{\text{sq}}(t) \propto \prod_{m < -\infty} \delta(t - mT/2)(-1)^m \]

Where \( \delta(t-a) \) is a \( \delta \)-function: equal to zero everywhere but at \( t=a \);

\[ \int_{t=a}^{t} \delta(t-a) \, dt = 1 \]

so that indeed

\[ \int_{t=a}^{t} V_{\text{diff}} \, dt \propto V_{\text{square}}(t) \]

Also, Fourier spectrum of such \( V_{\text{diff}} = \sum_{n} \frac{a_n}{\sqrt{\pi T}} V_{\text{diff}}(t) \cos(\omega_{0}nt) \, dt \propto 1 \) (const of \( n! \))

\[ b_n \propto \int_{-\Delta t}^{\Delta t} V_{\text{diff}}(t) \sin(\omega_{0}nt) \, dt = 0 \text{ nodes at } t=mT/2 \]

\[ V_{\text{diff}} \propto \sum_{n=1}^{\infty} \cos(\omega_{0}nt) \]

In a real differentiator, even if \( \omega_{0} \ll 1/\tau \) at some \( n \), \( n\omega_{0} \sim 1/\tau \) and these components do not get differentiated, keeping a \( 1/n \) amplitude dependence of a square wave.

The result is spikes of finite width (\( \sim RC \)).
Amplifiers.

Amps. could be represented by a four-terminal box, just like passive circuits: $V_{out}(s) = A(s) \cdot V_{in}(s)$.

The most important example is operational amplifier (OPAMP). It has a huge $\Delta V_{out} (>10^6)$ gain at low frequency:

$V_{out} = A(V_+ - V_-)$ which falls off at high frequency: $A \propto 1/s$

OPAMPs are used with negative feedback:

$V_+ = V_{out} \Rightarrow V_{out} = A(V_{in} - V_{out})$

$V_{out} = \frac{A}{A+1} \cdot V_{in} \approx V_{in}$ — "voltage follower"

This circuit is needed to combine the high impedance of the source $R_s$ with a low impedance of the load $R_L$. OPAMP's $R_{in} > 1M\Omega$ or higher, so it won't load the source ($R_s << R_{in}$), and its output $R_{out} < 10$ is low, so it won't be loaded by $R_c > R_{out}$.
OPAMP circuits

\[ V_{\text{out}} = A \left( V_{\text{in}} - V_{\text{out}}/2 \right) \]

\[ V_{\text{out}} = \frac{A}{A^2 + 1} V_{\text{in}} \approx 2V_{\text{in}} \]

This is the idea behind using OPAMPs: we reduce the gain, but the gain of the circuit is controlled by the resistors, rather than some nonlinear elements (see the actual OPAMP in §282)

For a general 4-terminal circuit in the feedback, \[ V_{\text{out}} = A(V_{\text{in}} - F(S)V_{\text{out}}) \]

\[ V_{\text{out}} = \frac{A V_{\text{in}}}{A F(S) + 1} \approx V_{\text{in}}/F(S) \] so for the whole circuit \[ H(S) = 1/F(S) \]

Real OPAMP needs to have power supplied to drive the load. Typically, these are both + and - supplied like this:

<table>
<thead>
<tr>
<th>+</th>
<th>-</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

Legs 1, 5, and 8 could be used for adjusting DC offset of \( V_{\text{out}} \).

Watch for leakage from 1 and 4 to 2 and 3!
Metal, Insulators, Semiconductors

Atomic levels in a crystal broaden into bands. Filled bands do not conduct current. Electrons per level result in a filled band. Partially filled bands appear if there is 1e per level (say Li) or bands overlap (group II metals). If all bands are completely filled or empty \( \Rightarrow \) insulator. Insulator with a small gap \( \Rightarrow \) semiconductor. In a SC, electrons can be thermally excited from a completely filled band (valence band) to a completely empty band (conduction band) resulting in a creation of charge carriers: \( e^- \) and \( h^+ \).

Carriers could be also added through doping. \( P(B) \) in Si has one e more (less) than the host, resulting in creation of \( e^- (h^+) \), donors/acceptors.
P-N Junction

If we connect n-doped and p-doped regions, charge will be transferred, resulting in build-up of $E$ (evidenced in band bending) until $E_F$ is aligned. Depletion region is created in the middle.

Diode

Naively, current through the p-n junction should be zero, until a voltage (almost) equal to the band gap is applied in the forward direction (the direction which flattens the bands).

Beyond this $V_{pn}$, $I = (V - V_{pn})/R_f$.

In ideal diode, $V_{pn} ightarrow 0$, $R_f ightarrow 0$.
Drift & diffusion

At zero applied voltage, there are equilibrium currents that exactly compensate each other. Let us consider e's only.

(Short p-n junction, electrons & holes do not recombine.) There is a flow of e's from n to p (majority carrier diffusion) and back from p to n (minority carrier drift).

At $V=0$, $I=I_D-I_M=0$

If $V \neq 0$ one assumes $I_D(V)=I_D(0) e^{eV/kT}$

$I_M(V)=I_M(0)$ - denote $I_0$. Then

$I(V)=I_0(e^{eV/kT}-1) \approx \frac{e I_0}{kT} V$

$V_T = \frac{kT}{e} = 30 \text{ mV}$ is often empirically replaced by $50 \text{ mV}$.

Also, $I_0 \propto$ concentration of minority carriers

$\alpha e^{-E_g/2kT} = 7\%/\degree C$ \quad (\frac{E_g}{2kT} \approx 15\% ),

is it $E_g/2$ ?)
Light Emitting Diodes (LED)

Electron and hole can annihilate (just like el. & positron!) emitting a photon with its equal to the band gap energy.
\[ I_c (I_B, V_{CE}) \Rightarrow \]
\[ V_{BE} (I_B, V_{CE}) \Rightarrow \]
\[ (1) \quad i_c = h_{fe} i_B + h_{oe} V_{CE} \]
\[ (2) \quad V_{BE} = h_{ie} i_B + h_{re} V_{CE} \]

\[ R_L \]

\[ h_{ie} \]
\[ h_{re} \]
\[ h_{fe} \]
\[ 10^{-4} \]
\[ 10^{-4} \]
\[ 10^{2} \]
\[ (10^{5} \Omega)^{-1} \]

\[ V_{CE} = -i_c R_L \]

Plug in (1) \[ i_c = \frac{h_{fe} i_B}{1 - h_{oe} R_L} \]

If \( R_L h_{oe} \ll 1 \)

\[ h_{oe} \sim (10^{5} \Omega)^{-1} \]

\[ I_{RL} \gg I_{\frac{1}{h_{oe}}} \]

\[ R_L \text{ should be } 10^{4} \Omega \]

\[ \Rightarrow \quad i_B = \frac{V_{BE} - h_{fe} V_{CE}}{h_{ie}} = \frac{V_{BE} + h_{re} h_{fe} R_L i_B}{h_{ie}} \]

\[ i_B = \frac{V_{BE}}{h_{ie} - h_{re} h_{fe} R_L} \]

\[ \frac{V_{CE}}{h_{ie} - h_{re} h_{fe} R_L} \approx \frac{R_L h_{fe}}{h_{ie} h_{re}} \]

\[ R_L \text{ voltage gain} \]

\[ \approx 10^{3} \Omega \text{ to } 10^{2} \Omega \]

\[ i_B \approx \frac{V_{BE}}{h_{ie}} (\text{V}) \]

\[ i_c \approx \frac{V_{BE}}{h_{fe}/h_{ie}} \text{ where } h_{fe}/h_{ie} \text{ is transconductance} \]
Perfect 
\[ i_c = h_{fe} i_B \]

No VON, since this is AC approximation.

Ideal tr-r: 
\[ h_{ie} = 0 \quad (V_{BE} = V_{CE}) \]

Common E. Amplifier.

Biasing
\[ I_B = \frac{V_{cc} - V_{PN}}{R_B} \]
\[ I_c = h_{FE} I_B \]
\[ V_c = V_{cc} - R_c I_c = V_{cc} \left(1 - \frac{R_c}{R_B} \frac{1}{h_{FE}}\right) \]

Should be \(-2V < V_c < V_{cc}\)

Conveniently, \(V_c \sim V_{cc}/2 \Rightarrow V = 1/2\)

\[ R_B = 2h_{FE} R_c \]
Not so good, as \(R_B\) depends on tr-r \((h_{FE})\)

Better
Base is tied to collector, not to Vcc.

\[ I_B \rightarrow \Rightarrow I_C \rightarrow \Rightarrow \]

\[ V_C = V_{cc} - I_C R_C \rightarrow \Rightarrow \]

\[ I_B = \frac{V_C}{R_F} \rightarrow \text{Negative feedback,} \]

which adds "stability" = Biasing stability.

Solving:

\[ V_C = V_{cc} - I_C R_C = V_{cc} - \frac{V_C}{R_F} \frac{h_{FE} R_C}{R_F + h_{FE} R_C} \]

For \( V_C = V_{cc}/2 \)

\[ R_F = \frac{h_{FE}}{R_C} \]

(Surprise! Half resistance for half voltage.)

Extra resistor base to ground could be added as a divider to reduce \( R_F \).

More on bias stabilization...

\[ V_B = V_E = R_E I_C = h_{FE} \frac{R_E I_B}{R_{UV}} \]

\[ B \overset{\text{RE}}{\text{hfe}} \]

Therefore, for the base part of the circuit, one can replace \( R_{UV} \) by \( \frac{R_{hfe}}{h_{fe}} \)

\[ V_B = \frac{R_{hfe}}{R_{hfe} + R_B} V_{cc} \]

\[ B \overset{\text{RE}}{\text{hfe}} \]

\[ V_E = V_B \Rightarrow I_C \approx I_E = \frac{V_E}{R_E} = \frac{h_{fe} V_{cc}}{R_{hfe} + R_B} \]

\[ \Rightarrow R_B \gg h_{fe} R_E \]

\[ \text{Cont} \rightarrow \]
\[ V_c = V_{cc} - R_c I_c = V_{cc} \left( 1 - \frac{h_{FE} R_c}{R_B + h_{FE} R_E} \right) \]

This circuit is also the Common Collector Amplifier, AKA Emitter Follower (just take \( V_c \) out)

AC signals
Short DC supply \( (V_{cc}) \) to ground:

\[ \frac{V_c}{V_B} = \frac{h_{FE} R_c}{h_{ie}} \]  
\[ \text{Input } Z \text{ is } h_{ie} || R_B \times h_{ie} \text{ for } R_s \neq 0 \]
\[ \frac{V_B}{V_S} = \frac{h_{ie}}{h_{ie} + R_s} \Rightarrow H = \frac{V_c}{V_S} = -\frac{h_{FE} R_c}{h_{ie} + R_s} \]
\( i_F = \frac{(V_B - V_C)}{R_F} \)  

\[ R_F i_F = h_{ie} i_B + R_C (h_{fe} i_B - i_F) \]

\[ i_F = \frac{h_{ie} + R_C h_{fe}}{R_F + R_C} i_B \]

\[ i_S = i_B + i_F = \frac{h_{ie} + R_F + R_C (h_{fe} + 1)}{R_F + R_C} \]

Input impedance: \( R_i = \frac{V_B}{i_B + i_F} = \frac{h_{ie} (R_F + R_C)}{h_{ie} + R_F + R_C h_{fe}} \)

Limits: \( R_F \to \infty \) \( R_i \to h_{ie} \) \( R_F \to 0 \) \( R_i \to h_{ie} / h_{fe} \)

Miller effect: reduction of \( R_i \) by the "gain" w/o feedback ("Open loop gain")

\[ (3) \Rightarrow V_C = -R_C (h_{fe} i_B - \frac{h_{ie} + R_C h_{fe}}{R_F + R_C} i_B) = -R_C \frac{h_{fe} (R_F + R_C) - h_{ie} - R_C h_{fe}}{R_F + R_C} i_B \]

\[ A = \frac{V_C}{V_B} = \frac{R_C (h_{ie} - R_C h_{fe})}{h_{ie} (R_F + R_C)} \]

Limits: \( R_F \to \infty \) \( A = -\frac{R_C h_{fe}}{h_{ie}} \)

\( R_F \to 0 \) \( A = 1 \)
Capacitive Feedback

Replace $R_F$ with $1/C_F$:

$A = \frac{-R_{chfe}/C_F + R_{chfe}}{h_{ie}(R_c + 1/C_F)}$

$= \frac{-R_{chfe}/h_{ie}(R_cC_F + 1)}{h_{ie}R_cC_F + 1}$

$Z_m = \frac{h_{ie}(R_c + 1/C_F)}{h_{ie}R_cC_F + 1}$

$= \frac{h_{ie}R_cC_F + 1}{h_{ie}R_cC_F + 1}$

$S_1 = \frac{1}{h_{ie}R_cC_F} = \frac{1}{h_{ie}AC_F}$

Miller effect

$\text{i}_{RF} = \text{V}_{in} - \text{V}_{out} = \text{V}_{in} + A\text{V}_{in} \Rightarrow Z_m = \frac{\text{V}_{in} = \frac{RF}{A+1}}{A}$

Feedback impedance in an inverting amplifier is seen as $Z_m$ to the ground, where $Z_m$ is $-RF/A \ll RF$.

In the previous case, $A = R_{chfe}/h_{ie}$, so

$Z_m = 1/C_F A = \frac{h_{ie}}{R_{chfe}C_F}$. At $s_1$, $Z_m = h_{ie}$ (at which $s$ is $Z_m$ becoming smaller than $h_{ie}$), so

$S_1 = \frac{1}{R_{chfe}C_F}$, just as we got from the detailed solution.
CE amplifier with $R_E$

$$V_B = i_B h_{ie} + i_B (h_{fe} + 1) R_E \Rightarrow i_B = \frac{V_B}{h_{ie} + R_E h_{fe}}$$

$$V_C = -R_{c} i_C = -\frac{V_B R_{chfe}}{h_{ie} + R_E h_{fe}} \approx -R_c \frac{V_B}{R_E}$$

Adding $R_E$ reduces gain, compared to $R_{chfe}/h_{ie}$ but makes it independent of variations of $h$'s.

Also, $R_{in} = (h_{ie} + R_E h_{fe}) || R_E$ is greatly increased.

To increase gain at high frequency, add $C_E$. Then
$$A = \frac{h_{fe} R_c}{(h_{ie} + h_{fe} Z_E)}$$

where $Z_E = R_E || \frac{1}{\frac{1}{C_E}}$. There are 2 corner frequencies as $\frac{1}{C_E}$ becomes smaller then $R_E$ and then $h_{fe}/C_E$ goes smaller than $h_{ie}$

Just like w/o $R_E, C_E$. 

$A = \frac{R_c}{R_{cE}} \frac{1}{R_{chfe}/h_{ie} C_E}$
**Junction Field Effect Transistor (JFET)**

Growing depletion region in the reverse biased P-N junctions pinches the channel as \( V_{DS} \) grows.

Two regimes: 1) variable resistor

\[ I_D \propto V_{DS} \text{ controlled by } V_{GS} \text{ and } V_{DS} \]

2) pinch-off. PD starts at the drain, where the voltage difference to the gate is the largest. As \( V_{DS} \) grows, it propagates, while \( I_D \) stays a const.

\[ I_D = I_{DSS} \left( 1 - \frac{V_{GS}}{V_0} \right)^2 \]

\( I_{DSS} \) - saturated \( I_D \) with gate shorted to source.

**Biasing (self)**

\( R_G \) is only needed to provide a pass for a tiny \( I_G \) (pA range). Thus \( V_G = 0 \), \( V_{GS} = V_G - V_S = -V_S = -I_G R_S = -I_D R_S \), determine \( R_S \):

\[ I_D = I_{DSS} \left( 1 - \frac{I_D R_S}{V_0} \right)^2 \Rightarrow R_S = \frac{V_0}{I_D} \left( 1 - \sqrt{\frac{I_D}{I_{DSS}}} \right) \]
Small signal model

\[ I_D = I_D (V_{GS}, V_{DS}) \quad (V_{GS} \text{ instead of } I_B) \]

\[ i_D = \frac{\partial I_D}{\partial V_{GS}} V_{GS} + \frac{\partial I_D}{\partial V_{DS}} V_{DS} \]

Neglect \( \frac{\partial I_D}{\partial V_{DS}} \) (curves are flat in pinch off)

\[ i_D = g_m V_{GS} \quad \text{where } g_m = \frac{\partial I_D}{\partial V_{GS}} \quad \text{is transconductance} \]

Typically a few \( \frac{1}{kR} \)

Common source amp.

\[ V_S = g_m V_{GS} R_S = g_m (V_G - V_S) R_S \]

\[ V_S = \frac{g_m R_S}{g_m R_S + 1} V_G \]

(Could make a follower)

\[ V_D = -i_D R_D = -\frac{V_S}{R_S} R_D = -\frac{g_m R_D V_G}{g_m R_S + 1} \]

If higher AC gain is desired, add \( C_S \parallel R_S \). Then

\[ A = \frac{R_D}{R_S \parallel C_S + \frac{1}{g_m}} \]

\[ R_D g_m \quad \frac{2\pi}{R_S} \quad \frac{1}{\kappa C_S g_m} \]
High Frequency

Add $C_GD$ & $C_GS$. Of these, $C_GD$ is more important, & it is magnified by the Miller effect.

$$\text{Take } R_S = 0$$

Solve: Current balance at drain:

$$\frac{V_G - V_D}{Z_{GD}} - \frac{V_D}{R_D} - g_m V_G = 0 \Rightarrow A = \frac{V_D}{V_G} = \frac{Z_{GD} - 1}{Z_{GD} + R_D}$$

If $g_m R_D \gg 1$, there are two corner frequencies. The second one is not reached, i.e.:

$$|A| = \frac{1}{g_m R_D}$$

$$V_G = \frac{Z_{GD}}{R_{in} + Z_{DG}/A} = \frac{Z_{DG}}{A R_{in} + Z_{DG}}$$

with a corner at $S^* = \frac{1}{R_{in} R_D g_m C_{DG}}$. Plug in $A = R_D g_m$:

$$S^* = \frac{1}{R_{in} R_D g_m C_{DG}}$$

typically $< S_2$ and perhaps $S_1$?

$$V_S, S_2: \quad g_m R_D R_{in} \gg 1$$

$$V_S, S_1: \quad g_m R_{in} \gg 1$$

RF amplifier—one can compensate for decreasing gain due to capacitors by adding $L_{\text{drain}}$ ($L_{DS} \rightarrow L$)

One can also tune the frequency band by replacing $R_D$ with a tank circuit ($L_{11C}$).
\[ V_{\text{in}} - V_{g} = \frac{V_{g}}{Z_{1} + Z_{2}} \]

Then \[ \frac{V_{D}}{V_{\text{in}}} = \frac{-Z_{1}Z_{2}g}{Z_{1}Z_{2} + \text{R}_{\text{in}} + \text{R}_{\text{in}}Z_{2}g} \]

For a high frequency JFET, \( gZ_{1} \gg 1 \) for \( f < 10^{6}Hz \)

If \( Z_{2} = \frac{1}{S_{C_{6}}} \)

and \( Z_{2} = L \parallel C \) then \( Z_{1} \parallel Z_{2} = L \parallel (C \parallel C_{6}) \)

\[ \frac{V_{D}}{V_{g}} = \frac{-Z_{1}Z_{2}g}{Z_{1}Z_{2}} \] - resonance gain at \( \omega \) = \( \frac{1}{\sqrt{L(C+C_{6})}} \) tunable by \( C \)
Inverting amplifier

\[ I = \frac{\text{Vin}}{R} \quad \text{Vout} = -IR_F = -\text{Vin} \frac{R_F}{R} \]

The same circuit w/o \( R \) can be viewed as a current amplifier (current-to-voltage converter).

It has a zero input resistance (finite \( I \) with \( \emptyset \) input voltage) — "virtual ground".

If \( A \) is finite: \( \text{Vout} = -AV_\text{in} \), \( I = (V_- - V_{\text{out}})/R_F \)

\[(A+1)V_- = IR_F \Rightarrow \text{input impedance} = R_F/(A+1)\]

Important at high \( \omega \) when \( A \) drops.

Current summing junction

\[ \text{Vout} = -R_F \left( \frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3} \right) \]

All currents add at the virtual ground and flow into \( R_F \).

Open loop gain

Dominant pole amplifier

\[ A = \frac{A_0}{1 + i\omega/\omega_0} \]
Integrator

\[ I = \frac{V_{\text{in}}}{R}, \quad V_{\text{out}} = -\frac{\Omega}{C} = -\frac{1}{RC} \int V_{\text{in}} \, dt \]

Since the gain cannot go above \( A_0 \), at low frequencies ideal gain \( G = \frac{1}{RC} \) is cut by \( A_0 \).

To prevent divergence due to DC \( R_f \) (very large) is added

"leaky integrator"

Differentiator

\[ I = \frac{dQ}{dt} = C \frac{dV_{\text{in}}}{dt}, \quad V_{\text{out}} = -\frac{RC}{A} \frac{dV}{dt} \]

At finite \( \omega \), \( G \) intersects with \( A \) and the actual gain \( H \) is limited.

Behavior at \( G \& A \) intersection could be a problem. Analyze:

\[ I_{\text{in}} = (V_{\text{in}} - V_{\text{in}}) \frac{1}{Z_c} = (V_{\text{in}} - V_{\text{out}}) \frac{1}{R}, \quad V_{\text{out}} = -AV_{\text{in}} \Rightarrow \]

\[ V_{\text{out}} = -\frac{V_{\text{in}}}{R} \frac{1}{Z_c + \frac{R + Z_c}{A}} = -V_{\text{in}} R \frac{1}{Z_c + RA_0(1 + \frac{s}{\omega_0})} \]

Looking for poles:

\[ \frac{R}{A_0} C S (1 + \frac{s}{\omega_0}) + 1 = 0 \]

\[ s^2 + \omega_0 s + A_0 \omega_0 / RC = 0 \]

Usually, 2nd term is small

\[ s_0 = \pm i \sqrt{A_0 \omega_0 / RC} \]

\( \text{GBP} \geq \text{MHz} \)

\( s \)

\[ s_0 \text{ such that } 1 / RC s_0 \leq S_0 \]

\[ \text{Poles have } R \leq A_0 \]

Circuit rings or even generates at \( s_0 \)
Differential amplifier

Purpose: To measure the voltage difference \( V_1 - V_2 \) between two points, rejecting common-mode voltage \( (V_1 + V_2) / 2 \). Solving: \( V_+ = \frac{R_2}{R_1 + R_2} V_2 \)

Since \( V_- = V_+ \):

\[
\frac{V_1 - V_+}{R_1} = \frac{V_+ - V_{out}}{R_2}
\]

\[
V_{out} = \frac{1}{R_1} \left[ V_+(R_1 + R_2) - V_1R_2 \right] = \frac{R_2}{R_1} (V_1 - V_2)
\]

Disadvantage: finite input resistance, input current in \( I \) depends on \( V_2 \). (How about CMRR?)

Instrumentation amplifier

3 op-Amp Inst Amp resolves these problems:

Think: w/o \( R_1 \), two front amps simply buffer the inputs; with \( R_1 \):

\[
\frac{V_1 - V_2}{R_1} = \frac{V_3 - V_4}{2R_2 + R_1}
\]

\[
V_{out} = \frac{2R_2 + R_1}{R_1} \frac{R_4}{R_3} \frac{V_2 - V_1}{V_2 - V_1}
\]

All \( R \) pairs are matched, except for \( R_1 \), which is user-selected.
Negative impedance converter:

\[ I_{\text{in}} = \frac{V_{\text{in}} - V_{\text{out}}}{R} \]

Use: \( R_3 \) compensates \( R_1 \) in the resonant circuit.

Gyrator:

\[ I_s = V_{\text{in}}/Z_5, \quad I_y = I_s \Rightarrow V_{34} = V_{\text{in}} + \frac{Z_4}{Z_5} V_{\text{in}}/Z_5 \]

\[ I_3 = (V_{\text{in}} - V_{34})/Z_3 = -V_{\text{in}} Z_4/Z_3 Z_5 \]

\[ I_2 = I_3 \Rightarrow V_{12} = V_{\text{in}} - V_{\text{in}} Z_2 Z_4/Z_3 Z_5 \]

\[ I_{\text{in}} = (V_{\text{in}} - V_{12})/Z_1 = V_{\text{in}} Z_2 Z_4/Z_1 Z_3 Z_5 \]

Mimics various elements. E.g. if \( Z_2 = \frac{1}{\omega C} \):

\[ Z_{\text{in}} = \frac{Z_1 Z_3 Z_5}{Z_2 Z_4} = \frac{\omega C R^2}{Z_2 Z_4} = \text{inductor!} \]

If \( Z_1 = Z_3 = Z_0 \), \( Z_{\text{in}} = \frac{1}{Z_0 R C^2} = \text{double capacitor} \)

Using multiplier:

\[ V_{\text{out}} \times V_2 \text{ should be } = V_1 \Rightarrow V_{\text{out}} = V_1/V_2 \]

\[ (V_{\text{out}})^2 \text{ should be } = V_{\text{in}} \Rightarrow V_{\text{out}} = \sqrt{V_{\text{in}}} \]
Effects of nonidealities:

\[ R_{in} \neq 0, \ R_{out} \neq 0, \ I_{bias}, \ I_{offset}, \ V_{offset}, \ A \neq \infty \]

Input impedance \( R_T \)

\[ I = \frac{V_{in}}{R_T} + \frac{(V_{in} - V_{out})}{R_F}, \ V_{nt} = -AV_{in} \]

\[ I = V_{in} \left[ \frac{1}{R_T} + \frac{(A+1)}{R_F} \right] \]

\[ Z_{in} = R_T \parallel \frac{R_F}{A+1} \]

Usually \( Z_{in} = R_F/(A+1) \) which shunts \( R_T \).

Noninverting. Ideally, \( A \gg G = \frac{R_1 + R_2}{R_1} \)

Since \( V_{out} = GV_{in} \) and also \( V_{out} = AV_+ - V_- \)

\[ V_+ - V_- = \frac{G}{A} V_{in}, \ I_0 = \frac{(V_+ - V_-)}{R_T} = \frac{G}{A} V_{in}/R_T \]

\[ Z_{in} = \frac{RA}{G} \gg R_T. \]

Output impedance

Let \( R_0 \) be the open loop output impedance.

To find the equivalent impedance for a closed loop system.

Take \( Z_{out} = V_{out} \parallel I_{out} \). Open circuit \( V_{out} = V_1 = A(V_{in} - V_{out}) \)

\[ V_{out} = \frac{AV_{in}}{A+1} \]

Short circuit: \( V = V_{out} \parallel 0, \ V = AV_{in} \)

\[ I_{out} = AV_{in}/R_0, \ \Rightarrow Z_{out} = \frac{R_0}{A+1} \]

\[ Z_{out} = \frac{R_0(\omega^2 + \omega)}{(A_0 + 1)\omega^2 + \omega} \]

\[ \text{There appears an effective inductance \( R_0/\omega \lambda \)} \]

If the load is capacitive, there will be resonant gain at

\[ \omega = \sqrt{\frac{\omega_0 R_0}{C_{load}}} \]

Related knowledge: slew rate \( \frac{V}{\text{rate (V/sec)}} \)

at which the amp's output approaches a step.
Stability analysis

\[ H_{inv} = \frac{A}{1+AF}, \quad H_{inv} = -\frac{A}{1+(A+1)F} \quad \text{where} \quad F \]

is the transfer function of the feedback network. Instabilities may occur at the poles of \( H \), when \( AF+1 = 0 \). Often, \( A \approx (s_0/s)^n \) at high frequencies, where \( A \approx 1/F \). Then \( F(s_0/s)^n = 1 \). \( S^* = \sqrt{F} \cdot s_0 e^{i\pi n} \) is OK for \( n = 1 \), marginal for \( n = 2 \), unstable for \( n > 3 \), (5). If \( F(w) \neq \text{const} \), the dangerous case is \( F \propto 1/s \) (like \( F = RCs \)).
\[ \langle V^2 \rangle = \int_0^\infty J_+(\omega)\,d\omega = \int_0^\infty e_n^2\,df \quad \Rightarrow \quad e_n^2 = 4kTR \]

\[ e_n = \sqrt{e_n^2} \approx 120\,\text{nV}/\sqrt{\text{Hz}} \quad \text{for 1 M\,\Omega} \]

\[ \text{in} = \frac{e_n}{R} = 120\,\text{fA}/\sqrt{\text{Hz}} \]

\[ \text{real} \leftrightarrow \text{en ideal} \leftrightarrow \text{ideal} \]

Example: \( R_1 || R_2 \)

\[ e^2 = \left( e_1^2 + e_2^2 \right) \left( R_1 || R_2 \right)^2 = 4kT \left( \frac{1}{R_1} + \frac{1}{R_2} \right) R_1 R_2 R_1 R_2 = 4kT \left( R_1 || R_2 \right)^2 \]

\[ \text{noises are added incoherently, } \frac{1}{2} \langle i_1 i_2 \rangle = 0 \]

\[ e^2 = e_1^2 \left( \frac{R_2}{R_1 + R_2} \right)^2 + e_2^2 \left( \frac{R_1}{R_1 + R_2} \right)^2 = 4kT \frac{R_1 R_2^2 + R_2 R_1^2}{(R_1 + R_2)^2} = 4kT \left( R_1 || R_2 \right)^2 \]

Alternatively, \( e_1 \) sends current through a \( R_2 / (R_1 + R_2) \) divider, or vice versa.

More noises: Shot noise

Shot noise is observed when there are discrete electron passing events (vacuum lamps, tunneling diodes).

\[ I = \langle i \rangle = N \left( \frac{e}{h} \right) \]

\[ \langle i^2 \rangle = N\left( \frac{e^2}{h} \right) = \frac{I^2 e^2}{h} \quad \text{where} \quad I = \frac{e}{h} = \frac{e}{h} - \text{Noise}^2 \text{ grows linearly with increasing } BW \times 1/\nu_o. \]

\[ \langle i^2 \rangle = K_i(\omega) = \int J_+(\omega)\,d\omega = \int e_n^2\,df \]

\[ i_n^2 \propto I e_i \] Exactly: \[ \langle i^2 \rangle = 2I_i e_i df \]

1/f Noise (pink, telegraph...)

\[ e_n^2 \propto 1/f \quad \text{Due to switching. Important at low } f \]

\( \langle e_n^2 \rangle \text{ formally diverges} \)
Noise in OP AMP

The noise is modeled by 3 sources: $e_n$, $i_{\text{in}}$, $i_{\text{m}}$

$e_n$ may not really be present at the input, but could be referred to input ($\frac{e_n}{A}$)

It is important to consider the full circuit.

Calculate $e_{\text{out}}$: Noise voltage on the $+$ leg:

$$e_{R_3} + (R_3i_{\text{in}})^2 + e_n$$

It gets multiplied by the gain ($\frac{R_1 + R_2}{R_1}$, the noise gain)

Contribution of $i_{R_1}$, $i_{R_2}$, $i_{\text{m}}$ to $e_{\text{out}}$ can be calculated separately, assuming $V_+ = 0 \Rightarrow V_- = 0 \Rightarrow$ all currents flow through $R_2 \Rightarrow (e_{\text{out}})^2 = (i_{R_1}^2 + i_{R_2}^2 + i_{\text{m}}^2) R_2^2$. To refer it back to input, one divides $e_{\text{out}}$ by the noise gain resulting in $\left(i_{R_1}^2 + i_{R_2}^2 + i_{\text{m}}^2\right) \left(\frac{R_2^2}{R_1 + R_2}\right)^2$.

Combining:

$$\left(e_{\text{in}}^2\right) = e_n^2 + e_{R_3}^2 + (R_3 i_{\text{in}})^2 + e_{R_1 i R_2}^2 + (R_1 i R_2)^2 i_{\text{in}}^2$$

assuming $i_{\text{in}} = i_{\text{m}} = i_{R_1}$

$$\left(e_{\text{in}}^2\right) = e_n^2 + 4kT \left(R_1 i R_2 + R_3\right) + i_{\text{m}}^2 \left[R_1 i R_2 + R_3\right]^2$$

$r \approx e_n/i_{\text{m}}$ characteristic noise resistance. For a good amplifier, in a wide range of $R$ around $e_n$,

$$4kT R \gg e_n^\frac{2}{r_{\text{in}}^2}$$
More complicated gates

\[ AB + CD \quad \text{- AND-OR gate} \quad (\text{either } A \text{ and } B \text{ or } C \text{ and } D \text{ are true}) \]

\[ \overline{AB} + A\overline{B} = (A + B)(\overline{A} \cdot \overline{B}) \quad \text{- EOR = XOR, exclusive or gate} \]

\begin{align*}
000 & \quad \text{good} \\
101 & \quad \text{or} \\
110 & \quad \text{bad! due to glitches}
\end{align*}

Example of a circuit with asynchronous timing - requires time diagrams. Alternatively, use synchronous (clocked) systems.

Adders

Add 2-bit \( X_1X_0 \) and \( Y_1Y_0 \) : \( X_0 + Y_0 = C_1 \cdot C_0 \)

\[ Z_0 = \overline{X_0}Y_0 + X_0\overline{Y_0} \quad , \quad C_1 = X_0Y_0 \]

For the second bit, there are 3 inputs : \( X_1, Y_1, \) and \( C_1 \)

\[ C_2 = \overline{C_1}X \cdot Y_1 + C_1\overline{X} \cdot Y_1 + C_1X \cdot \overline{Y_1} + C_1X \cdot Y_1 = \]

\[ X_1Y_1 + C_1Y_1 + C_1X \]

with the first 3 terms

\[ Z_1 = C_1 \oplus (X \oplus Y_1) \quad \text{carry} \]

[Diagram of adder circuit]
Binary, octal, hexadecimal numbers

1 Byte = 8 Bit = 2 hexadecimal digits (00 to FF)
0 to 255_{10} or alternatively -128_{10} ÷ 127_{10}

Boolean Algebra

Variables have 2 values: 0 & 1

Algebraic operations - \(\text{AND}: \quad \overline{R} = A \cdot B\)

Also \(\text{NOT}: \quad \overline{Q} = \overline{A}\)

\(\text{OR}: \quad Q = A + B\)

Most operations work as we are used to:

\(A \cdot 0 = 0, \quad A + 0 = A, \quad A \cdot 1 = A\)
\(A + 1 = 1, \quad A \cdot \overline{A} = 0, \quad A + \overline{A} = 1\)
\(A \cdot B = B \cdot A, \quad A + B = B + A, \quad A + (B + C) = (A + B) + C\)

\(A \cdot (B + C) = A \cdot B + A \cdot C\) — De Morgan’s theorem:

\(\overline{A \cdot B} = \overline{A} + \overline{B}\)

\(\overline{A + B} = \overline{A} \cdot \overline{B}\)

Logical gates

\(\text{AND:} \quad A \cdot B\)

\(\text{NAND:} \quad \overline{A \cdot B}\)

\(\text{OR:} \quad A + B\)

\(\text{NOR:} \quad \overline{A + B}\)

\(\text{inverter:} \quad \overline{A}\)

Truth table:

<p>| | | | | | |</p>
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<thead>
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These gates are redundant:

e.g. by De Morgan’s theorem

\(\overline{A \cdot B} = A + B\)

\(\overline{A + B} = \overline{A} \cdot \overline{B}\)
RSFF

\[
\begin{array}{cccc}
S & R & Q\bar{Q} & \text{no ch} \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
\end{array}
\]

avoid

\[
\begin{array}{cccc}
S & R & Q\bar{Q} & \text{NCH} \\
0 & 0 & 11 & \text{avoid} \\
0 & 1 & 10 & \text{avoid} \\
1 & 0 & 01 & \text{avoid} \\
1 & 1 & \text{NCH} & \text{avoid ambiguous of 11 input} \\
\end{array}
\]

DFF

\[
\begin{array}{cccc}
D & C & Q\bar{Q} & \text{NCH} \\
0 & 0 & 01 & \text{avoid} \\
1 & 1 & 10 & \text{avoid} \\
\end{array}
\]

JKFF

\[
\begin{array}{cccc}
J & K & C & Q\bar{Q} \\
0 & 0 & 0 & \text{NCH} \\
0 & 1 & 0 & 01 \\
1 & 0 & 1 & 0 \\
1 & 1 & \text{toggle} & \text{avoid ambiguous of 11 input} \\
\end{array}
\]
Counters

count enable

count
count