1 . <u>Projective Representations</u>

connects to anomalies.

<u>Wigner's Theorem</u>

If $\sigma : \mathcal{H} \to \mathcal{H}$ is a Hilbert space map representing a physical symmetry

then : • $\sigma(v+w) = \sigma(v) + \sigma(w)$

• $\sigma(av) = \chi(a)v$ $\quad a \in \mathbb{C}$.

• $\langle \sigma(v), \sigma(w) \rangle = \chi(\langle v, w \rangle)$

where $\chi(a) = a$ for all a "unitary"

or $\chi(a) = \bar{a}$ " "antiunitary".

If the Hamiltonian is preserved then antiunitary is taken to unitary by time reversal. We only consider unitary.

Symmetries are really important:

~~If~~ $g(t) \in \mathcal{U}(\mathcal{H})$ is a "one param. subgroup"

(t not nec. time!)

$g(0) = 1$.

$g(s+t) = g(s)g(t)$

<u>Stone's Theorem</u> If $g(t)$ is a one-param subgroup of $\mathcal{U}(\mathcal{H})$ which is "strongly continuous" then

$$g(t) = \exp(itA)$$

for some self-adjoint op A.

($g(t)^{-1} H g(t) = H \iff [A, H] = 0$.)

So symmetries and conserved ~~quantal~~ observables are related!

2. But wait - we asserted $U(\mathcal{H})$ too soon!

Wigner's theorem says if the symmetry group is G then there is a <u>map</u> $g: G \to U(\mathcal{H})$
but noone said it was a <u>hom</u>.

States are <u>rays</u> in the Hilbert space
- normalized states ~~and~~ are vectors
up to mult. by $c \in U(1)$
$|c| = 1$.

Let $PU(\mathcal{H}) = \dfrac{U(\mathcal{H})}{U(1)}$ <u>assume</u> $\mathcal{H}$ finite-dim'l.

A symmetry group ~~is really a hom~~ G has a hom
$$g: G \to PU(\mathcal{H})$$

hom ~~If~~ $g: G \to GL(V)$ is a "linear rep"
$g: G \to PGL(V)$ is a "projective rep."

Since there is a quotient map $U(\mathcal{H}) \to PU(\mathcal{H})$
any linear rep is projective - but not v.v.

<u>Prop</u> There is a surjective hom
$$\pi: SU(n) \to PU(n)$$
$\ker(\pi) = \mathbb{Z}_n$ gen by $I e^{2\pi i/n}$.

Since $\ker(\pi)$ is finite, this is an isomorphism of Lie algebras. $su(n) \cong pu(n)$.

3. So reps of $su(n)$ and $pu(n)$ are the same.

~~Prop~~ ~~Any~~ If H has Lie alg $\mathfrak{h}$
G " " $\mathfrak{g}$ and G is simply-conn.
then any hom
$$f_*: \mathfrak{g} \to \mathfrak{h}$$
is induced from a map
$$f: G \to H.$$

So any projective rep. $G \to PU(n)$
lifts to a linear rep. $\tilde{G} \to SU(n)$
where $\tilde{G}$ is the universal cover of G.

So we just replace G by $\tilde{G}$.
E.g. $SO(3)$ replaced by $SU(2)$ – hence spinors!
Any rep. of $su(2)$ is a proj. rep of $SO(3)$.

But this is fishy when $n \to \infty$.

4. More general approach:

If $\rho: G \to U(\mathcal{H})$ then we know

$$\rho(gh) = \varepsilon(g,h)\rho(g)\rho(h)$$

for some $\varepsilon(g,h) \in U(1)$.

ρ is associative, which yields

$$\varepsilon(g_1, g_2 g_3)\,\varepsilon(g_2, g_3) = \varepsilon(g_1, g_2)\,\varepsilon(g_1 g_2, g_3)$$

Note, if $\rho(1) = 1$ then $\varepsilon(1,g) = 1$

Define $\hat{G}$ from ρ and G as follows: $\hat{G} = U(1) \times G$ as a set.

If $(c,g) \in U(1) \times G$, then define

$$(c_1,g_1)\cdot(c_2,g_2) = (\varepsilon(g_1,g_2)^{-1} c_1 c_2,\ g_1 g_2)$$

and $\rho': \hat{G} \to U(\mathcal{H})$

by $\rho'(c,g) = c\rho(g)$.

Then ρ' is indeed a homomorphism.

Note that $1 \longrightarrow U(1) \xrightarrow{i} \hat{G} \xrightarrow{q} G \to 1$.

is a short exact sequence of groups.

where $i(c) = (c, \cancel{\rho(1)}\,1)$ (we fix $\rho(1) = 1$)

$q(c,g) = g$.

and $i\,U(1)$ is in center of $\hat{G}$.

$\hat{G}$ is a "central extension" of G.

So proj reps of G are linear reps of $\hat{G}$.

(Note $\hat{G}$ depends on ρ!)

5. If G is a Lie group, we can study this locally:

$$0 \to \mathbb{R} \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{q} \mathfrak{g} \to 0.$$

Central extension of Lie algebras.

As vector spaces, $\hat{\mathfrak{g}} \cong \mathbb{R} \oplus \mathfrak{g}$

but not as Lie algebras.

So define a ~~set map~~ map of vector spaces

$$\sigma : \mathfrak{g} \to \hat{\mathfrak{g}}$$

so that $q \circ \sigma = 1$.

and $[\sigma(X), \sigma(Y)] = \sigma([X,Y]) + f(X,Y)$

for some $\cancel{f \in \mathfrak{g}^* \otimes \mathfrak{g}^*}$ $f : \mathfrak{g} \times \mathfrak{g} \to \hat{\mathfrak{g}}$.

but

$$\begin{aligned}
[q\sigma(X), q\sigma(Y)] &= [X,Y] \\
&= q\sigma([X,Y]) + qf(X,Y) \\
&= [X,Y] + qf(X,Y)
\end{aligned}$$

so $qf = 0 \Rightarrow f : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$.

Clearly $f \in \Lambda^2 \mathfrak{g}^*$

Recall Lie algebra cohomology:

$$df \in \Lambda^3 \mathfrak{g}^*$$

$$df(X,Y,Z) = -f([X,Y],Z) - \text{cyclic}.$$

In this case one can show $df = 0$.

Suppose we replace σ by $\sigma' : \mathfrak{g} \to \hat{\mathfrak{g}}$
where $\sigma = \sigma' + ik$, $k : \mathfrak{g} \to \mathbb{R}$, $k \in \Lambda^1 \mathfrak{g}^*$
then $f(X, Y) = f'(X, Y) + k([X, Y])$
i.e. $f = f' + dk$ in Lie alg. cohom. lang.

~~so~~ This is an isomorphism of extensions
so extensions classified by $H^2(\mathfrak{g}, \mathbb{R})$.

Easiest example:

$$0 \to \mathbb{R} \xrightarrow{i} \mathfrak{g} \to \mathbb{R} \oplus \mathbb{R} \to 0$$

$[\,,\,] = 0$

Z. $\qquad [X, Y] = 0$

$f(X, Y) = 1 \qquad i(1) = i$

$X = \hat{x}$, $Y = \hat{p}$. (But this is just confusing!)
Heisenberg

Theorem (Bargmann).
If $H^2(\mathfrak{g}) = 0$ then every projective rep. of G can be lifted to a linear rep. of $\tilde{G}$.

Theorem (Whitehead)
If G is a finite-dim'l simple Lie group then $H^2(\mathfrak{g}) = 0$.

7. E.g. Virasoro Algebra: life on the circle S^1.

$\sigma = 0..2\pi$ parametrizes circle

Let σ' be a function of σ

- reparametrization

- C^∞
- invertible (also C^∞)
- fixed direction of circle

I.e. $\sigma' \in \mathrm{Diff}(S^1)$ ... group of diffeomorphisms

Consider $\mathrm{diff}(S^1)$ .. the Lie algebra.

$$\sigma \to \sigma + \varepsilon f(\sigma) \qquad \varepsilon \text{ very small}$$

$f(\sigma)$ periodic so use Fourier modes

$$\sigma \to \sigma + i\varepsilon e^{in\sigma} \qquad (\text{pass to } \mathbb{C}$$

- can fix afterwards

Consider rep. on functions $g: S^1 \to \mathbb{C}$

$$g(\sigma) \to g(\sigma + i\varepsilon e^{in\sigma})$$

$$= g(\sigma) + i\varepsilon e^{in\sigma}\frac{d}{d\sigma}g(\sigma) + \dots$$

so Lie alg. represented by

$$L_n = i e^{in\sigma}\frac{d}{d\sigma} \qquad n \in \mathbb{Z}.$$

so that change in $g(\sigma)$ given to first order by $1 + \varepsilon L_n$.

$$L_m L_n g(\sigma) \; \cancel{\text{So } [L_m, L_n]} = -e^{im\sigma}\frac{d}{d\sigma}e^{in\sigma}\frac{d}{d\sigma}g(\sigma)$$

$$= -in e^{i(m+n)\sigma}\frac{d}{d\sigma}g(\sigma) - e^{i(m+n)\sigma}\frac{d^2 g}{d\sigma^2}$$

$$= -n L_{m+n} g(\sigma) - e^{i(m+n)\sigma}\frac{d^2}{d\sigma^2}g(\sigma)$$

$$\Rightarrow \quad [L_m, L_n] = (m-n)L_{m+n}.$$

8. Note than by Stone's theorem these correspond to self-adjoint operators

i.e. $L_n^\dagger = L_{-n}$ from Fourier decomp.

This is diff(S^1) .. an infinite dim'l algebra.

Note L_{-1}, L_0, L_1 form subalgebra $sl(2,\mathbb{C})$ (but not ideal.)

<u>Theorem</u> $H^2(\mathrm{diff}(S^1), \mathbb{R}) = \mathbb{R}$.

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9. Apply to a vibrating string: 

- creation operators for modes α_n $n < 0$
- ann " " α_n $n > 0$

centre of mass momentum α_0.

Can easily obtain...

$$x = \bar{x} + \cancel{\alpha_0 t}\ \alpha_0(\sigma + t) + i \sum_{n \neq 0} \frac{1}{n} \alpha_n e^{-in(\sigma + t)}$$

for left-moving waves.

$$\underline{[\alpha_m, \alpha_n] = m\,\delta_{m+n}}$$

About as easy as it could be (must have odd power of m)

$$\partial x = \sum_n \alpha_n e^{-in(\sigma+t)}$$

α_n are Fourier modes of ∂x.

If A is an operator that transforms as

$$A'(\sigma') = \left(\frac{d\sigma}{d\sigma'}\right)^h A(\sigma)$$

(A is h^{th} power of tangent bundle of S^1)

then, if $A(\sigma) = \sum_n A_n e^{-in\sigma}$

$$[L_m, A_n] = (m(h-1) - n)\, A_{m+n}$$

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So $h = 1$ for ∂x gives

$$[L_m, \alpha_n] = -n\,\alpha_{m+n}.$$

10. Hilbert space of vibrating string is Fock space
- should write L_m in terms of α's.

$$\Rightarrow \quad L_m = \frac{1}{2}\sum_k \alpha_{m+k}\alpha_{-k}$$

$[L_m \alpha_n]$, $[L_m, L_n]$ work.

except that

$$L_0|0\rangle = \frac{1}{2}\sum_{k\geq 1} k\,|0\rangle \qquad \text{ouch!}$$

Fix by normal ordering

$$:\alpha_m \alpha_n: = \begin{cases} \alpha_m \alpha_n & m < n \\ \alpha_n \alpha_m & m > n. \end{cases}$$

$[L_m, \alpha_n]$ unaffected

$[L_m, L_n]$ gets $c = 1$!

Fermion: Easiest commute is $\{a_m, a_n\} = \delta_{m+n}$.

$h = \frac{1}{2}$.

So $[L_m, a_n] = (-\frac{1}{2}m - n)\, a_{m+n}$.

$$\Rightarrow \quad L_m = \frac{1}{2}\sum_r r :a_{-r} a_{m+r}:$$

- Easiest thing it can be.

then $c = 1/2$.