PART II - BRST

Poisson Algebra: Algebra of functions & Poisson bracket $\{ , \}$.

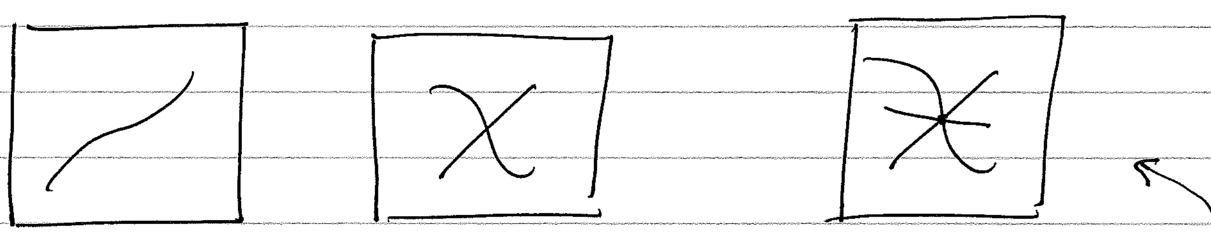
Set of constraints φ_i

Restrict the algebra to a new one

Subquotient - Quotient out by I given by φ_i

Restrict to function invariant under flow given by φ_i
(Action of gauge group.)

The # of φ_i's I need may not be the same as dim of gauge group action.



1 function → hypersurface

2 functions

eg 3 functions s.t. $\boxed{\text{PROBLEM}}$

Koszul complex

$$\cdots \to \Lambda^2 \mathfrak{g} \otimes C^\infty(M) \to \mathfrak{g} \otimes C^\infty(M) \to C^\infty(M) \to 0$$

What does it take to be able to quantize?
Replace ~~commutator~~ P.B. w/ commutator
"constraints better not be identically zero"
In order to quantize Poisson algebra must get rid of constraints φ_i.
Only want nondegenerate Poisson bracket.

E.g $\{q_i, p^j\} = \delta_i^j \quad p^2 = 0 \quad \Sigma$
Can turn into nondegenerate Poisson algebra + constraints.

K for Koszul

BACK TO KOSZUL COMPLEX

$$\Lambda^2 \mathfrak{g} \otimes C^\infty(M) \xrightarrow{\delta_K} \mathfrak{g} \otimes C^\infty(M) \xrightarrow{\delta_K} C^\infty(M) \to 0$$

$\mathfrak{g}$ is some vector space w/ bases X_i
$X_i \sim \varphi_i$'s

$$\delta_K (f \cdot X_{i_1} \wedge X_{i_2} \wedge \ldots \wedge X_{i_k})$$
$$= \sum_{m=+1}^{k} (-1)^m f \varphi_{i_m} X_{i_1} \wedge \ldots \wedge \hat{X}_{i_m} \ldots \wedge X_{i_k}$$

$$\delta_K^2 = 0$$

H^0 is $C^\infty(M) / \mathrm{Im}\, \delta_K = C^\infty(M) / I$

$H^k = 0$ for $k > 0$ if we have complete intersection, no relations btwn φ_i's.

From now on call X_i "b_i"

Elements of koszul complex are polynomials in coordinates of M, b_i's

(in part. elements of $C^\infty(M) \otimes \Lambda^\circ \mathfrak{g}$)

b/c of Jacobi identity on PB

~~Polynomials are antisymmetric~~ $\mathfrak{g}$ lie algebra but may be ridiculous.

Lie Algebra Cohomology

Let G be a lie group, define left multiplication

$$l_a : G \to G \quad l_a(g) = ag \quad \text{for } a \in G$$

This map pulls back differential forms

We say ω is left invariant if

$$l_a^* \omega = \omega \quad \forall a \in G$$

Then similarly we can have left inv vector fields.

$\mathfrak{g}$ is the Lie algebra of left invariant vector fields.

The space of left-invariant p forms is therefore $\Lambda^p \mathfrak{g}^*$

If ω is left invariant then so is $d\omega$.

i.e we have

$$d : \Lambda^p \mathfrak{g}^* \longrightarrow \Lambda^{p+1} \mathfrak{g}^*$$

If $f \in \Lambda^0 \mathfrak{g}^*$ i.e a function, then its left inv if its a const $\Rightarrow df = 0$.

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$$\mathcal{L}_V w = i_V dw + d\, i_V w$$

Lie derivative — contraction

Given a v.f V

Let V & W be vector fields on G, assume both are left invariant. Assume ω is a left invariant 1-form

Then

$$i_V(w) = i_W(w) = \text{const}$$

$$\text{so } d\, i_V(\omega) = 0$$

$$\& \quad \mathcal{L}_V(i_W(w)) = 0 = \mathcal{L}_V(w(W))$$

$$= (\mathcal{L}_V \omega)(W) + \omega(\mathcal{L}_V(W))$$

$$= (i_V dw)(W) + \omega([V,W])$$

$$= dw(V,W) + \omega([V,W])$$

$\mathcal{L}_V W = [V,W]$ constructed using right action on V, otherwise $\mathcal{L}_V W = 0$

So $dw(V,W) = -\omega([V,W])$.

Similarly $d\omega(U,V,W) = -\omega([U,V],W) + \omega([U,W],V) - \omega([V,W],U)$ for ω a 2-form.

For an n-form

$$d\omega(V_1, \ldots, V_{n+1}) = \sum_{i<j} (-1)^{i+j} \omega([V_i, V_j], V_1, V_2, \ldots, \hat{V}_i, \ldots, \hat{V}_j, \ldots, V_{n+1})$$

We now define Lie algebra cohomology as

$$0 \to \Lambda^0 \mathfrak{g}^* \to \Lambda \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^* \to \cdots \to \Lambda^p \mathfrak{g}^* \to \Lambda^{p+1} \mathfrak{g}^* \to \cdots$$

($d^2 = 0$ follows just from Jacobi so we could have defined all of this w/o lie group but it's good for motivation).

[Note that if G is compact, then lie algebra cohomology is de Rham cohomology of G.]

"If bracket is trivial (abelian lie group) & compact have cohomology of a torus."

More generally let G act on M ie some G orbit on M

$$\text{Then,} \quad d\omega(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i \underbrace{X_i}_{\text{action of PB}}(\omega(X_0, \ldots, \hat{X}_i, \ldots X_k))$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], \ldots, \hat{X}_i, \hat{X}_j \ldots, X_k)$$

X_i's a push forward elements of $\mathfrak{g}$

ω is any k-form on M.

$X_i(f)$ is $\{\varphi_i, f\}$ action of PB.

Then we have another cochain complex

$$\Lambda^\bullet \mathfrak{g}^* \otimes C^\infty(M)$$

Note that $H^0 = \mathfrak{g}$ inv functions.
these are forms on orbit of G.

$$X_i \cdot (\omega(X_0, \dots, \hat{X_i}, \dots, X_k))$$

G action.

NEXT TIME WILL COMBINE INTO A DOUBLE COMPLEX!!