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BRST YAAAAAS!

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References:
Notes by Figeroa - O'Faril
Notes by Holten hep-th/0201124
Main result from today due to Nasheff
"Constrained Poisson Algebras & strong homotopy representations"
Henneaux + Teitelbaum (hard to read)

The goal is to quantize a system that has gauge symmetry
Two approaches to QM: • take phase space + canonical quantization relations $[f,g] = i\hbar\{f,g\}_{PB}$
• take path integral

SIMPLEST EX:
Consider a Lagrangian $L = \frac{1}{2}\dot{q}_1^2$, q_1 & q_2 are cood's. (canonical coods)

$$p^1 = \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1$$

$$p^2 = \frac{\partial L}{\partial \dot{q}_2} = 0$$

Key statement: L is invariant under 2 symmetries

$q_1^* \longrightarrow q_1^* + c$ This called a global symmetry

$q_2 \longrightarrow q_2 + f(t) \longleftarrow$ any function of time
This is called a local symmetry

WANT $[q_2, p^2] = \hbar$ PROBLEM: we had $p^2 = 0$
we need to fix this!

3 possible sol'ns:

(1) Get rid of stupid symmetry! Get rid of q_2.
can't always do this.

(2) "Fixing the gauge" by breaking symmetry

(3) Add more stuff to "cancel symmetry"

We'll do (3).

Phase space is q_1, q_2, p^1, p^2
To remove the gauge symmetry, we need to fix $p^2 = 0$ (this is a constraint)
there's no natural # to fix it to so we quotient out by shifts in q_2.
What remains is phase space q_1 & p^1
Call it a SUBQUOTIENT.

often in Gauge theory – have more parameters than necessary but often hard to get rid off.

Phase space is a symplectic mfld M w/
symplectic form

$$\omega = dq_1 \wedge dp^1 + dq_2 \wedge dp^2$$

Could also say that the space of functions $C^\infty(M)$ has a Poisson bracket

$$\{f, g\}_{PB} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$$

More generally, we have <u>Noethers thm</u>

If a Lagrangian has a symmetry, then

1) If symmetry is global, we have a conserved <u>quantity</u> (in time subject to eqns of motion)

In our example: $\dot{p}^1 = 0 \Rightarrow p^1$ is const wrt time

2) If symmetry is local, then we have a constraint

= $\exists$ function on phase space w/ fixed value

[e.g $p^2 = 0$] [See van Holten - nicest proof]

Let the constraints be denoted by φ_i

Using symplectic geometry:

If X is a tangent field on M preserving the symplectic form

$\mathcal{L}_X \omega = \omega$ Liouville specializes to hamiltonian

~~($\Rightarrow \mathcal{L}_X \omega = \omega$ - Liouville v.f ?)~~

(X is a Hamiltonian v.f depending on top of M)

then $i_X \omega$ is closed

locally $i_X \omega = dH$ of some function H.

H is the function constrained to be 0.

~~For types of symmetry have to worry about~~

If G is the gauge grap, then any $Y \in \mathfrak{g}$ induces a v.f on M.

So this means a map

$$\Phi : \mathfrak{g} \longrightarrow C^\infty(M)$$

$$Y \longrightarrow X \longrightarrow dH$$

b/c Y comes from $\mathfrak{g}$, X preserves ω. & then $i_X \omega = dH$.

So $\Phi \in \mathfrak{g}^* \otimes C^\infty(M)$ or

$\Phi : M \to \mathfrak{g}^*$ "Moment Map"

So the constraint is $\Phi = 0$

Let $M_0 = \Phi^{-1}(0) \subset M$

choosing 0 to any c Does it matter? Lets assume this is the right const.

Our aim is to form $\Phi^{-1}(0) / \text{Action of } G.$

this is called a Symplectic Quotient.

We can think algebraically (Algebraic Geometry)

Consider functions on M, $C^\infty(M)$

Given two mflds

$$\begin{tikzcd} M \arrow[rr, "F"] \arrow[dr, "f"'] & & N \arrow[dl, "g"] \\ & \mathbb{R} & \end{tikzcd}$$

F induces a map

$$\Rightarrow \quad C^\infty(N) \longrightarrow C^\infty(M)$$

Contravariant.

If $M \subseteq N$ $M \longrightarrow N \Rightarrow C^\infty(N) \to C^\infty(M)$ is a quotient

~~If N = quotient space of M~~

Quotient by the ideal of all functions vanishing ~~functions~~ on M.

Conversely if $N \longrightarrow M$ is a quotient by some group action G.

$\Rightarrow C^\infty(M) \subset C^\infty(N)$ are the G invariant functions

Instead of a symplectic mfld consider a <u>Poisson algebra.</u>

What is it? A commutative associative algebra with a <u>bracket</u>, which is antisymmetric & obeys Jacobi identity & is a derivation.

Poisson mfld: M mfld whose algebra of functions is a Poison algebra. e.g on symp M, $C^\infty(M)$ is a Poisson algebra.

$$\{f,g\} = -\{g,f\}$$

$$\{f,\{g,h\}\} + \text{cyclic} = 0$$

$$\{fg, h\} = f\{g,h\} + g\{f,h\}$$

We say an ideal $I \subset A$ is a 1^{st} class if

$$\{I, I\} \subset I$$

(in physics: FIRST CLASS CONSTRAINTS.)

It is not nec true $\{I, A\} \subset I$

A derivation corresponds to a tangent field (in alg geometry a ~~derivation~~ tangent field is defined as a derivation)

i.e $\{f, -\}$ maps A to A

If $(a_1, a_2, \ldots)$ vector $\Rightarrow$ $a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + \ldots$ derivation.

So if $F \in A$ we have a tangent field X_F where $X_F(f) = \{F, f\}$

The bridge btwn symplectic + poisson!

The problem algebraically is to find the subalgebra of A/I which is invariant under the "I-flows."

A/I "=" $M_0 = \Phi^{-1}(0)$

e.g I = functions where $p^2 = 0$,

The answer can be phrased in terms of cohomology.

KOSZUL COMPLEX

We want a complex whose 0th cohomology is

$C^\infty(M)/I$ (or A/I)

Let $\mathfrak{g}$ be the Lie algebra of symmetries

(or Same vector space where dim = # constraints)

Let X_i be a basis for $\mathfrak{g}$
the moment map takes $X_i \longrightarrow \varphi_i$
constraints that generate I

~~choose~~ Use this to define

$$\mathfrak{g} \otimes C^\infty(M) \xrightarrow{\delta_K} C^\infty(M) \longrightarrow 0$$

where $\delta_K(X_i) = \varphi_i \qquad X_i \in \mathfrak{g}$

$$\delta_K(X_i \otimes g) = \varphi_i \cdot g \qquad g \in C^\infty(M).$$

This can be extended (to the left) to a complex called the Koszul complex

$$\cdots \to \Lambda^2 \mathfrak{g} \otimes C^\infty(M) \xrightarrow{\delta_K} \mathfrak{g} \otimes C^\infty(M) \xrightarrow{\delta_K} C^\infty(M) \to 0$$

where $\Lambda^n \mathfrak{g}$ has a basis $X_{i_1} \wedge X_{i_2} \wedge X_{i_3} \wedge \ldots$

Define $\delta(f \otimes X_{i_1} \wedge X_{i_2} \wedge X_{i_3} \wedge \ldots \wedge X_{i_k})$
$$= \sum_{m=1}^{k} (-1)^m f \, \varphi_{i_m} X_{i_1} \wedge \ldots \wedge \hat{X}_{i_m} \wedge \ldots \wedge X_{i_k}$$

Now we have a complex,

$$H_0 = C^\infty(M) / \mathrm{Im}\,\delta_K$$

$$= C^\infty(M)/I$$

Is it an exact complex? May or may not be true.
This complex is exact for $H_i(\ )$ $i>0$
If $(\varphi_1, \varphi_2, \ldots)$ is a regular sequence
i.e. $I=0$ is a "complete intersection"

If codim $M_0 \subset M$ is k
& I is generated by $\varphi_1, \ldots, \varphi_k$

This doesn't have to be true. If I is not a complete intersection, we need to specify more constraints than the codim $= k$.
$\Rightarrow$ we have to describe relations btwn the constraints.

Then, the Koszul reso is replaced w/ Tate resolution (BV instead of BRST)

e.g. when you quantize supergravity.