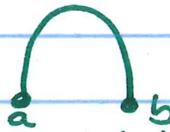
Goal: Construct a solution to YBE & a corresponding bracket polynomial.

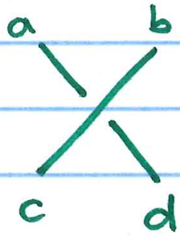
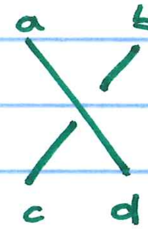
Toy Physics example.

t ↑  $\longleftrightarrow M_{ab}$ ⟵ tensor like objects

annihilation of spins a, b

t ↑  $\longleftrightarrow M^{ab}$

creation of spins a, b from vacuum

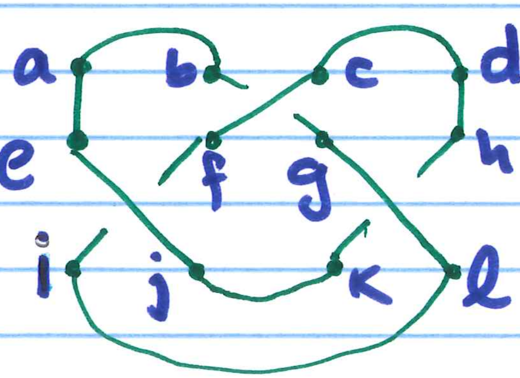
t ↑  $\longleftrightarrow R^{ab}_{cd}$  $\longleftrightarrow \bar{R}^{ab}_{cd}$

two particle interactions

if we want crossing symmetry $\Rightarrow \bar{R}^{ab}_{cd} = R^{ca}_{db}$

Then also

t ↑  $\longleftrightarrow \delta^a_b$ curves w/ no crit pts wrt height function.

EX 

$\longleftrightarrow T(K)$

$$= M_{ab} M_{cd} \delta^a_e \delta^d_h R^{bc}_{fg} \bar{R}^{ef}_{ij} \bar{R}^{gh}_{kl} M^{il} M^{jk}$$

tensors!

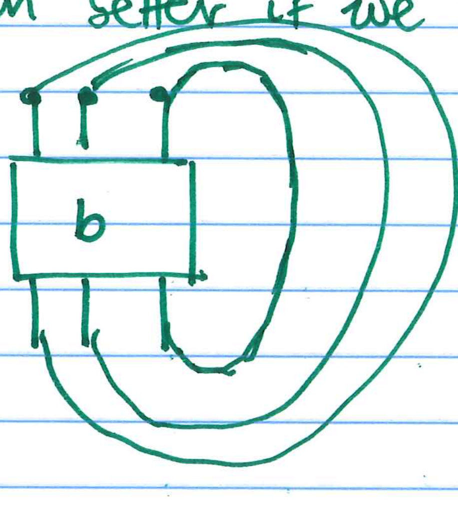
If these tensors M^{ab}, R^{ab}_{cd}, δ^a_b are numeraly valued (or valued in a commutative ring)
$\Rightarrow$ $T(K)$ is a vaccuum - vacuum expectation. for these process given by diagram & time arrow.

Sanity check: In QM the probability amplitude for the concatenation of processes is obtained by summing the products of the amplitudes of intermediate configurations in the process over all possible configurations

$$\sum_i P_{ai} Q_{ib} = (PQ)_{ab}$$

corresponds to matrix mult.

Even better if we consider a braid $\overline{b}$ closure

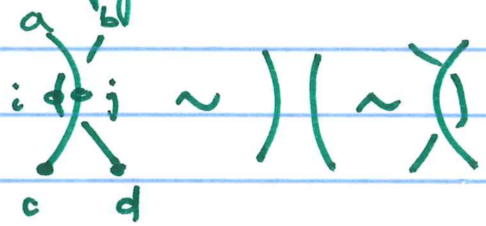


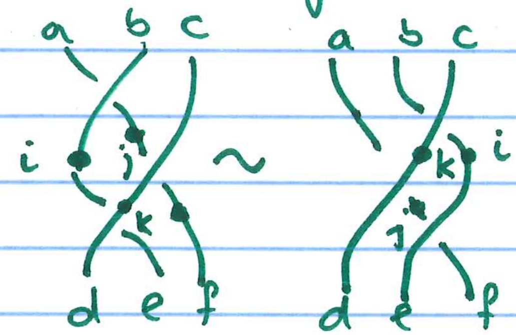
we have one max & min for each strand

$\leftrightarrow M_{ai} M^{bi} = \eta_a{}^b$

& only R-matrices on b. $\underset{K=\overline{b}}{T(K)} = \overbrace{\eta \cdots \eta}^{n \text{ of them}} \underbrace{[R\text{-matrices}]}_{@(b)}$

If we want $\tau(K)$ to be an invariant of isotopy of K, then it shouldn't change under





$$\bar{R}^{ab}_{ij} R^{ij}_{cd} = \delta^a_c \delta^b_d$$

so $R\bar{R} = I = R\bar{R}$

$$R^{ab}_{ij} R^{jc}_{kf} R^{ik}_{de} = R^{bc}_{ki} R^{ak}_{dj} R^{ji}_{ef}$$

YBE

[We won't bother with RI invariance for now b/c that can be fixed at the end.

Suppose we want

$\tau(K)$ to satisfy the bracket eqn

$$\tau(\times) = A\,\tau(\asymp) + A^{-1}\,\tau(\,)(\,)$$

$$R^{ab}_{cd} = A \;{}^{a}\!\smile\!{}^{b}_{c}\!\frown\!{}_{d} + A^{-1}\; {}^{a}_{c})({}^{b}_{d}$$

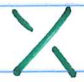
$$= A M^{ab} M_{cd} + A^{-1} \delta^a_c \delta^b_d$$

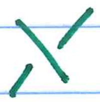
~~to get~~ If we also ask that

$$a \bigcirc b = \sum_{a,b} M_{ab} M^{ab} = d = -A^2 - A^{-2}$$

Then setting

$U = M^{ab} M_{cd}$ 

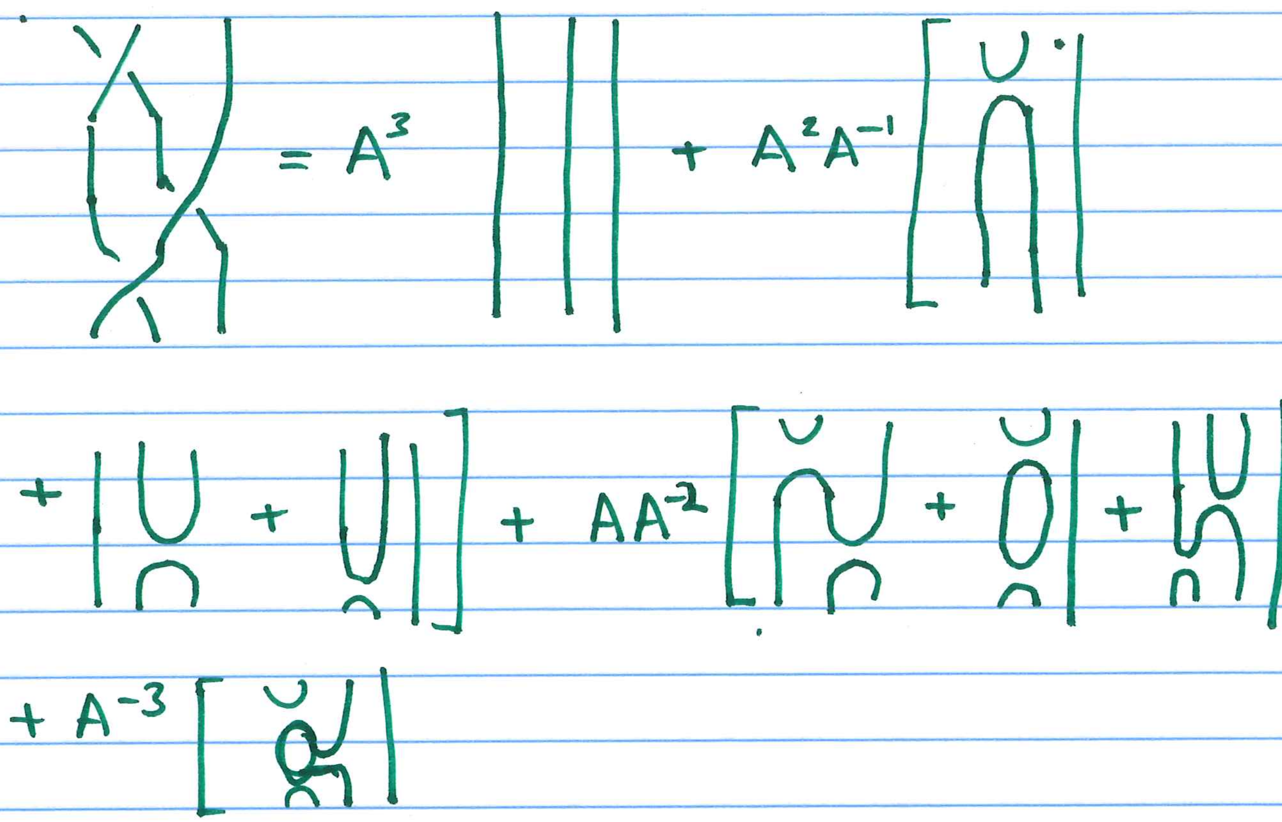
we have  $= AI + A^{-1}U$

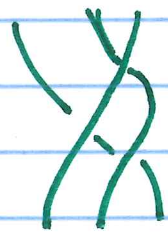
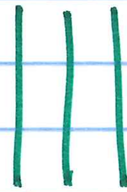
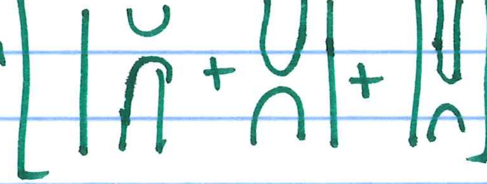
 $= AU + A^{-1})(= AU + A^{-1}Id$

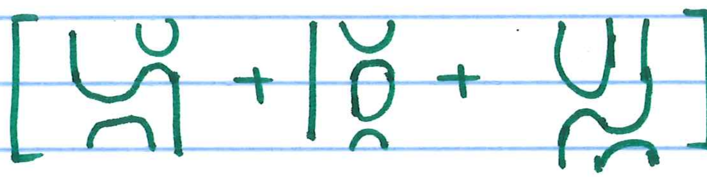
& $U^2 = dU$

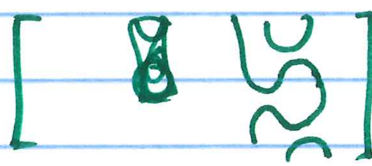
Then R satisfies the YBE.

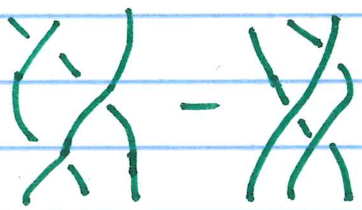
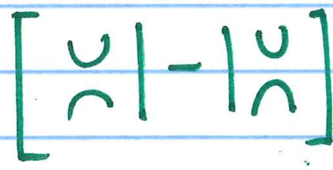
Check:

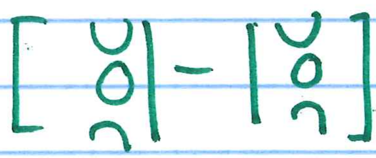
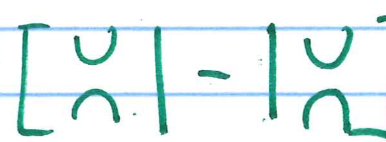


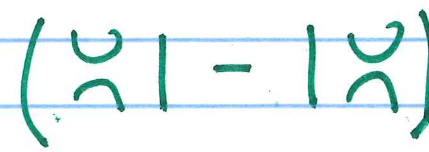
 $= A^3$  $+ A^2A^{-1}$ 

$+ AA^{-2}$ 

$+ A^{-3}$ 

So  $= A^2A^{-1}$ 

AA^{-2}  $+ A^{-3}$ 

$= \left[A^2A^{-1} + dAA^{-2} + A^{-3}\right]$ 

So we need

$A + (-A^2 - A^{-2})A^{-1} + A^{-3} = 0$

$A - A - A^{-3} + A^{-3} = 0$ ✓

So we need to get

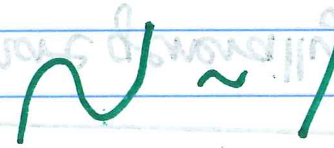
- A model for the bracket
- soln to YBN

is pair of matrices $M_{a,b}$ M^{ab} of inverse matrices s.t.

$$\sum_{a,b} M_{ab}M^{ab} = -A^2 - A^{-2}$$

<u>One solution</u>

Assume $M_{ab} = M^{ab}$

$\Rightarrow$  $\longleftrightarrow M^2 = I$

Let $M = \begin{pmatrix} 0 & iA \\ -iA^{-1} & 0 \end{pmatrix}$ $\quad M^2 = \mathrm{Id}$

$d = (-1)A^2 + (-1)A^{-2} =$

or we could pick

$$U = M \otimes M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -A^2 & 1 & 0 \\ 0 & 1 & -A^{-2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Why are we tensoring?

$$R = AM \otimes M + A^{-1} I \otimes I$$

$$R = \begin{pmatrix} A^{-1} & 0 & 0 & 0 \\ 0 & -A^3-A^{-1} & A & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & A^{-1} \end{pmatrix}$$

Note that

for each braid $b \longmapsto \underset{T(K)}{\underset{\shortparallel}{\langle b \rangle}} \subseteq \mathbb{Z}[A, A^{-1}]$ maybe.

b/c $T(K)$ has tensor summing over indices.

for $\boxed{A = \pm 1}$ ~~so~~ we have a representation of the braid grap

$$\rho : B_n \longrightarrow \cancel{\mathbb{Z}[A, A^{-1}]}\ \mathrm{Aut}((\mathbb{C}^2)^{\otimes n})$$

$$\sigma_i \to id^{i-1} \otimes R \otimes id^{n-i-1}$$

which ~~is also a~~ also gives a soln to YBE. (R)

R is a map from $\mathbb{C}^4 \to \mathbb{C}^4$.

More generally:

If V is a complex vector space, an automorphism R of $V \otimes V$

$$R : V \otimes V \longrightarrow V \otimes V$$ (an iso of $V\otimes V \to V \otimes V$)

satisfies the YBE if

$$(R \otimes id) \circ (id \otimes R) \circ (R \otimes id) = (id \otimes R) \circ (R \otimes id) \circ (id \otimes R)$$

R III.

Given such an R we can define a representation

$$\rho_R : B_n \longrightarrow \mathrm{Aut}(V^{\otimes n})$$

by $\sigma_i \longmapsto id^{i-1} \otimes R \otimes id^{n-i-1}$

Check if A representation: $B_n = \{\sigma_i, \sigma_i^{-1} \quad i = 1, \dots, n-1$ s.t. $\sigma_i\sigma_j = \sigma_j\sigma_i$ $|i-j|>1$; $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

$\cancel{\rho\rho\sigma_i\sigma_j}$

e.g. $j = i+2$

$$\rho_R(\sigma_i)\rho_R(\sigma_j) = (id^{i-1} \otimes R \otimes id^{n-i-1}) \cdot (id^{i+1} \otimes R \otimes id^{n-i-3})$$

$$\rho_R(\sigma_i)\rho(\sigma_{i+1})\rho(\sigma_i)$$

$$= (id^{i-1} \otimes R \otimes id^{n-i-1}) \circ (id^{i} \otimes R \otimes id^{n-i-2}) \circ (id^{i-1} \otimes R \otimes id^{n-i-1})$$

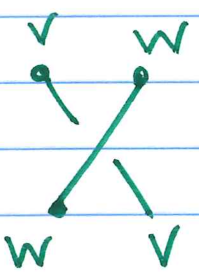
$$\overset{?}{=} \left[id^{i-1} \otimes (\overset{R\otimes 1}{\cancel{id \otimes R}}) \circ (id \otimes R)(R \otimes id) \otimes id^{n-i-2}\right]$$

$$= id^{i-1} \otimes (1 \otimes R) \cdot (R \otimes 1) \cdot (1 \otimes R) \otimes id^{n-i-2})$$

Final comment:

$\exists$ an object called a quasitriangular bialgebra
$(A, \mu, \eta, \Delta, \epsilon)$
whose category of modules is a braided monoidal category (More modern approach)

$\Rightarrow$ We get ~~solns~~ A map
isomorphism of A-modules
$$V \otimes W \longrightarrow W \otimes V$$



& from this we get a soln to the YB eqn

& a representation of braid group B_n.