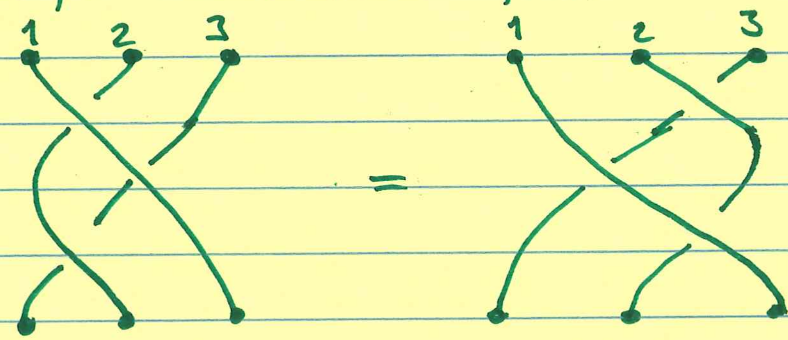
References: Kauffman Knots & Physics
Introduction to representations of braid groups
— Camilo Arias Abad
minicourse at Cimpa

I. Yang Baxter equation:

consistency equation introduced first in statistical mechanics (e.g 2D Ising Model)

"in some scattering situation, particles may preserve their momentum while changing quantum internal states"

"the interaction btwn 3 particles is determined by the two particle interactions & is independent of which particles interact first."



$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12} \qquad \text{YB eqn.}$$

R^{ij} is an endomorphism of $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ on the two factors i, j & leaving the third factor alone.

Key cool fact: Solutions of the YB eqn produce representations of the braid group.

~~These~~ Representations of the braid group are also related to knot invariants!

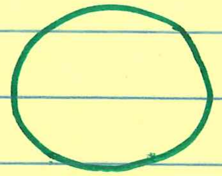
Goal: Sketch how a soln to YB eqn gives us the Jones polynomial. An important knot inv.

I. Background: KNOTS + Braids.

Defn A knot is an embedding $S^1 \hookrightarrow \mathbb{R}^3$ or $(S^1 \hookrightarrow S^3 = \mathbb{R}^3 \cup \{\infty\})$.

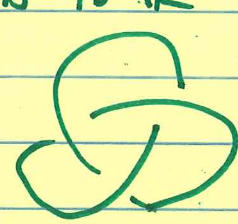
An n-component link is an embedding of n disjoint circles into $\mathbb{R}^3$

Projections of knots to $\mathbb{R}^2$

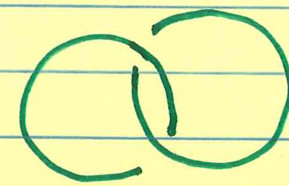


unknot

$\neq$



trefoil



Hopf link

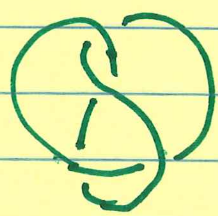


fig 8 knot.

We say two ~~knots~~ links K_1, K_2 are ~~ambient~~ isotopic if $\exists$ an ambient isotopy btwn them

This is a ~~map~~ conts map

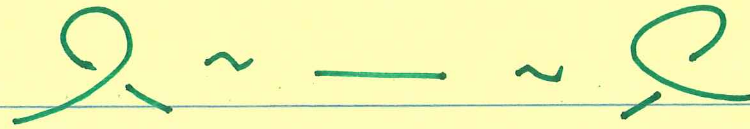
$$\phi_t : \mathbb{R}^3 \times I_t \longrightarrow \mathbb{R}^3$$

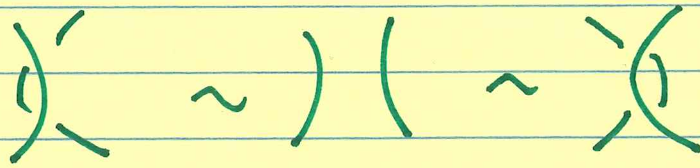
s.t. $\phi_0 = Id$ $\quad$ $\phi_t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ homeomorphism

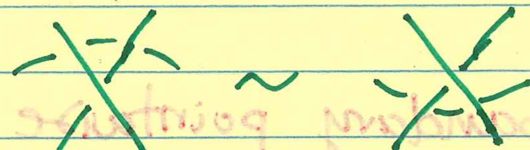
$\phi_1(K_1) = K_2$.

[Reidemeister] THM Two diagrams of links represent isotopic links if one can be transformed into the other by a finite sequence of 3 types of moves called Reidemeister moves

Fundamental group $\pi_1(X, x_0)$ loops in space X based at x_0 / homotopy

R I 

R II 

R III  (this is like what I drew for YB)

The question of how to distinguish non-isotopic knots was attacked with knot invariants (e.g minimal crossing number, seifert genus, $\pi_1(\mathbb{R}^3 \setminus K)$, polynomials, ...)

Thanks to reidemeisters thm, if you want to construct a knot inv from diagrams/projections of knots you just need to check that it doesn't change under Reidemeister moves.

homotopy: continuously deform one cont map to another.

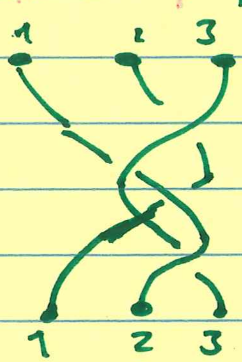
Braids:

Defn: Take n-pts on the plane, n-paths

$f_i : I \to D_2$

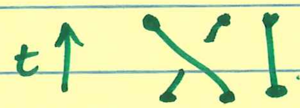
$f_i(0) = p_i \quad f_i(1) = p_{\tau(i)}$ ← permutation in S_n

$f_i(t) \neq f_j(t) \quad i \neq j$



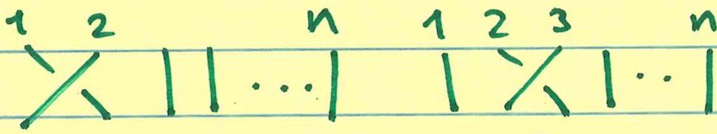
Note: Mapping Class group of the disk with n punctures (orientation preserving homeomorphisms of $\mathbb{D}^2$ that fix punctures as a set

Can think of braids as diagram of a physical process of partices interacting or moving about in the plane. where time is moving up vertically

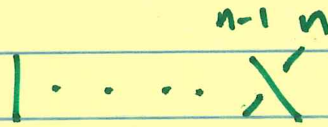


<u>Braid group B_n (n pts)</u>

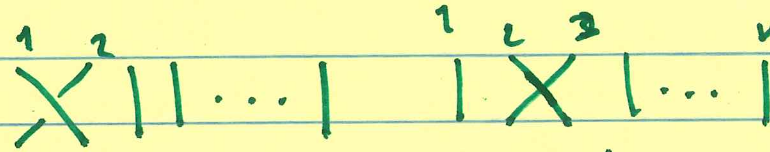
generators



σ_1 σ_2



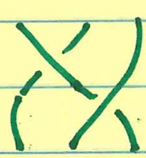
σ_{n-1}



σ_1^{-1} σ_2^{-1}

Then any braid $b \in B_n$ is a word

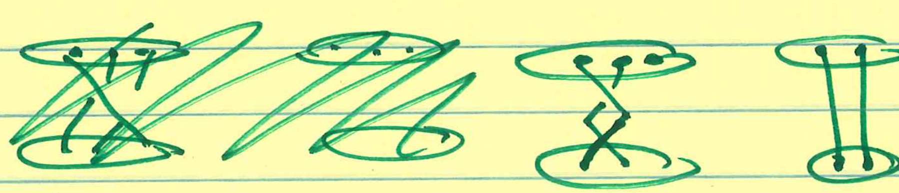
e.g



$\sigma_1^{-1}\sigma_2 = b$

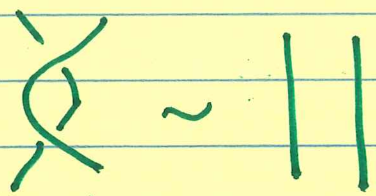
<u>Q</u> when are two braids $b, b' \in B_n$ equivalent

If there exists an ambient isotopy from b to b' that keeps endpts fixed & strands don't move abide top + bottom planes of braids.

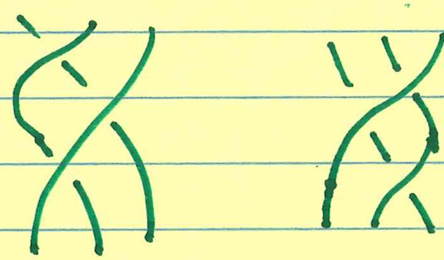


$i=1,\dots,n-1$

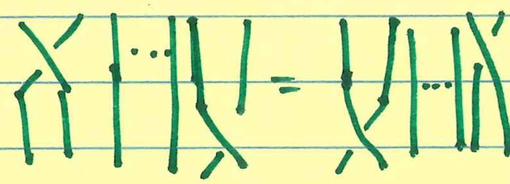
$$B_n = \langle \sigma_i, \sigma_i^{-1} \mid \begin{array}{ll} \sigma_i\sigma_i^{-1} = 1 & i=1,\dots,n-1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & i=1,\dots,n-2 \\ \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j|>1 \end{array} \Big\}$$



$\sigma_i\sigma_i^{-1} = 1$



$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$

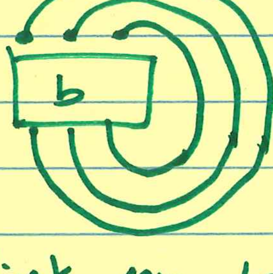


$\sigma_i\sigma_j = \sigma_j\sigma_i$

$|i-j|>1$

Q: How are braids related to knots?

[Alexander] Given a braid $b \in B_n$ its closure $\bar{b}$ is:



Any knot or link can be put in the form of a closed braid (via ambient isotopy)

Q How many different ways can a link be represented as a closed braid?

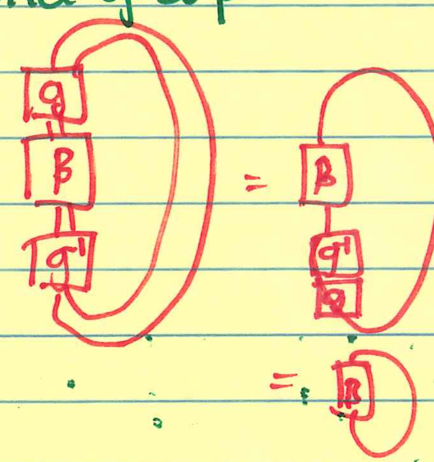
[MARKOV] $\beta_n \in B_n$, $\tilde{\beta}_m \in B_m$ two braids

$\overline{\beta_n} \sim \overline{\beta_m}$ as links iff $\tilde{\beta}_m$ can be obtained from β_n via

(1) Equivalences in a given braid group

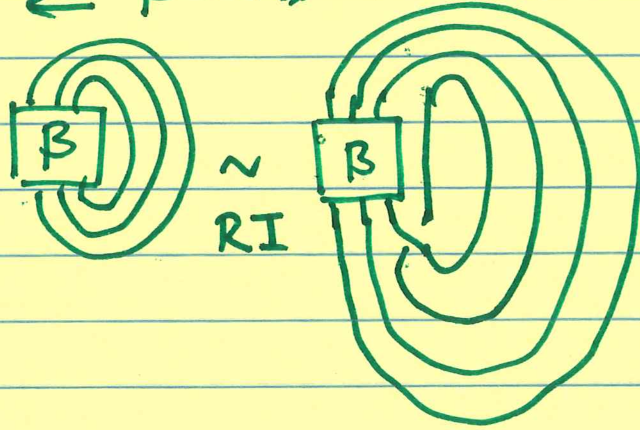
(2) conjugation

$$g\beta g^{-1} \sim \beta$$



(3) Markov moves

$$\beta \rightleftarrows \beta\sigma_n^{\pm 1} \in B_{n+1}$$



[Now: We can use __presentations__ of braid group B_n to extract topological info about knots + links.

Markov trace on B_n

$$J_n : B_n \to \boxed{R} \quad \text{ring } R$$

we need $J_n(b) = J_n(gbg^{-1})$ $\quad J_n(b) = J_n(\tilde{b})$

$$J_{n+1}(b\sigma_n) = \alpha^{+1} J_n(b)$$

↑ same const.

$$J_{n+1}(b\sigma_n^{-1}) = \alpha^{-1} J_n(b)$$

Then if
$$w(b) = \sum_{i=1}^{k} a_{\ell} \quad \text{for } b = \sigma_{i_1}^{a_1} \sigma_{i_2}^{a_2} \cdots \sigma_{i_k}^{a_k}$$

$$J(L) = \alpha^{-w(b)} J_n(b) \text{ is a link invariant}$$

($L = \bar{b}$) for $\{J_n\}$.

JONES POLYNOMIAL + BRACKET POLYNOMIAL.

More generally: V complex vector space, an automorphism R of $V \otimes V$ is an r-matrix if it satisfies the YB

$$(R \otimes id) \circ (id \otimes R) \circ (R \otimes id) = (id \otimes R) \circ (R \otimes id) \circ (id \otimes R) \in End(V^{\otimes 3}).$$

LEMMA: Given a r-matrix $R \in Aut(V \otimes V)$ $\exists$ representation

$$\rho_R : B_n \to Aut(V^{\otimes n})$$

$$\sigma_i \mapsto id^{i-1} \otimes R \otimes id^{n-i-1}$$

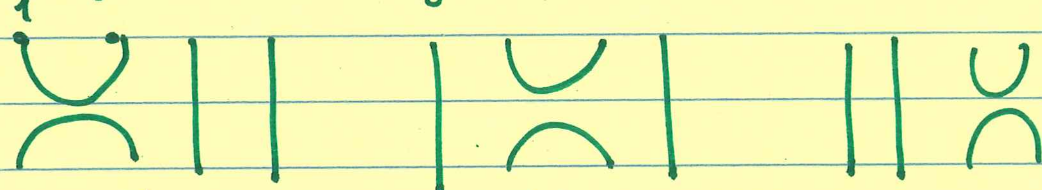
Pf

$\rho(\sigma_i \sigma_j) = \rho(\sigma_j \sigma_i)$ $|i-j|$ clearly works

$$\underbrace{\rho(\sigma_i)\rho(\sigma_{i+1})\rho(\sigma_i) = \rho(\sigma_{i+1})\rho(\sigma_i)\rho(\sigma_{i+1})}_{\text{by YB}}$$

Jones + bracket polynomial:

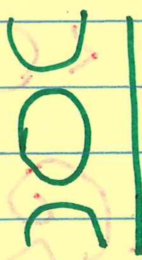
Consider the following Temperley-Lieb algebra ~~[illegible]~~ generated by U_i



U_1 $\quad$ U_2 $\quad$ U_3

Then $U_i U_{i\pm 1} U_i = U_i$

$U_i^2 = \delta U_i$



$U_i U_j = U_j U_i \quad |i-j| > 1$

Consider the following mapping

$$\rho(\sigma_i) = A + A^{-1} U_i$$

$$\langle || \cdots \times \cdots || \rangle = A \langle \underbrace{|| \cdots ||}_{Id} \rangle + A^{-1} \langle || \cdots \asymp \cdots || \rangle$$

$$\rho(\sigma_i^{-1}) = A^{-1} + AU_i$$

~~⇒ ρ: Bn → An → Z[A, A⁻¹] tr~~

~~by setting any loop = δ = δU.~~

~~⇒ ρ(b) = ⟨b̄⟩ is called the bracket polynomial~~

Now consider $tr : A_n \longrightarrow \mathbb{Z}[A, A^{-1}]$ where

$tr(U) = \langle U \rangle = \delta^{\|U\|}$

the number of closed loops

⇒ $\langle b \rangle = tr(\rho(b))$ & we have defined the bracket polynomial.

Note that

$$\langle b \rangle = \sum_{s} \underbrace{\langle b|s \rangle}_{\text{weights } A,\, A^{-1}} \underbrace{\langle \overline{U}_s \rangle}_{=\,\delta^{\|\overline{U}_s\|-1}}$$

state

~~⟨ ⟩ = A⟨ ⟩ + A⁻¹⟨ ⟩~~

~~= A⟨ ⟩~~

for $\langle \quad \rangle = \langle\ )(\ \rangle$ RII

⇒ $\delta = (-A^2 - A^{-2})$

& $\langle \quad \rangle = \langle \quad \rangle$ RIII $\qquad (A+A^{-1})(A^{-2}-A^2)$

But $\langle \quad \rangle = A\langle\ —\ \rangle + A^{-1}\langle \quad \rangle$

~~= A⟨ ⟩ + A⁻¹⟨ ⟩~~

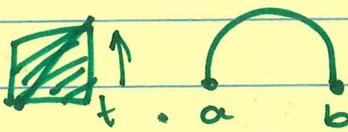
So we add a normalizing facter $(-A^3)^{-w(\beta)}$

$w(\beta)$ = writhe of a braid.

Then if we let $A = t^{-1/4}$ we get the Jones polynomial. which is a knot invariant

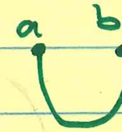
Jones is a specialization of the ~~bracke~~ trace of a rep of B_n.

Toy Physics example:



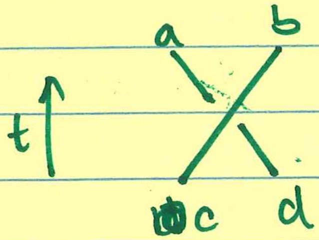
$\longleftrightarrow M_{ab}$

annihilation

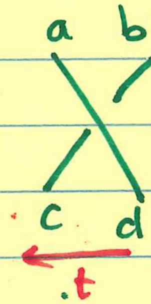


$\longleftrightarrow M^{ab}_{\cancel{ab}}$

creation



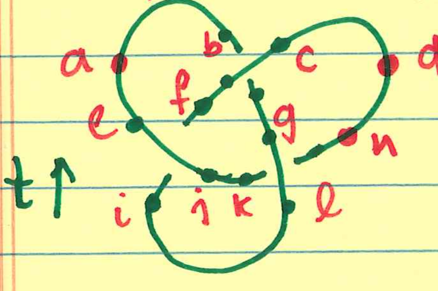
$\longleftrightarrow R^{ab}_{cd}$



$\longleftrightarrow \bar{R}^{ab}_{cd}$

they are differentiated by time + ~~but~~ could be diff interactions or

if we ask for $\bar{R}^{ab}_{cd} = R^{ca}_{db}$



$T(K)$ = M's & R's

$= M_{ab} M_{cd} R^{bc}_{fg} \ldots$

$= M_{ab} M_{cd} \delta_e^a \delta_n^a R^{bc}_{fg} \bar{R}^{ef}_{ij}$

$\bar{R}^{gn}_{kj} M^{il} M^{jk}$

$T(K)$ vacuum to vacuum expectation.

In a highly simplified QFT of link diagrams.

We want $T(K)$ to be a topological inv of reg isotopy (moves II + III)