PART IV: MATT : INDEX THM : 26/03/2018.

Last time: we talked about Clifford actions on vector bundles,

~~Given a cotangent vector~~, $c(e_i)$ acts on E_p for $e_i \in T_p^*M$

$E \longrightarrow M$ vector bundle.

We wanted $c(e_1)c(e_2) + c(e_2)c(e_1) = -2\langle e_1, e_2\rangle$

$e_i \in T_p^*M$.

We also defined the Dirac operator associated to c

$$D = \sum_i c(e_i)\nabla_i$$

where e_i is an orthonormal frame

well defined globally.

A few examples:

$\Lambda^\bullet T^*M$ has clifford action

$$c(e)\alpha = e \wedge \alpha - i_e \alpha$$

here e is mapped to vector in order to do contraction i_e using metric

$$D = d + d^*$$

Δ spin representation

$c(e_i)$ were given γ-matrices

local description: use triv of

$\mathbb{C}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4$

Physics land:

sigma model w/ M as a target space. (field space)

~~Nonlinear supermodel~~

Fermions take values on $TM \simeq T^*M$

~~Spa~~ + ...

Clifford module is a vector space with a Clifford action.

Δ is the "unique irreducible Clifford representation" in dim 4.

I mean that any (complex) Clifford module is of the form $\mathbb{C}^N \otimes \Delta$.

In particular, in dim 4,

$$\dim(\Lambda^\bullet T^*M) = 16$$

So $$\Lambda^\bullet T_p^*M = \mathbb{C}^4 \otimes \Delta$$

- Can we turn Δ into a bundle? (What happens when we vary p?)

For M^4 orientable

- Can we find an $SU(4)$ bundle $\boxed{\text{with a Clifford action}}$ $\Delta \longrightarrow M^4$ such that we can find local orthonormal frames of T^*M & trivializations of Δ such that locally the Clifford action is just $c(e_i)$ given γ-matrices ?

Δ spin representation

In general, no.

Say $U, \tilde{U} \subset M$ open over which we have oriented orthonormal frames $\{e_i\}$ & $\{\tilde{e}_i\}$

Then $\tilde{e}_i = \sigma_{ij} e_j$ for some $[\sigma_{ij}] \in SO(4)$.

wrt trivialization over U,

$$c(\tilde{e}_i) = \sigma_{ij}\, c(e_j)$$

Call $c(e_i)$ γ matrices γ_j

Then $c(\tilde{e}_i) = \sigma_{ij}\, c(e_j) = \sigma_{ij}\gamma_j$

wrt trivialization over $\tilde{U}$

$$c(\tilde{e}_i) = \gamma_i$$

So $\exists\ g \in SU(4)$

$$\gamma_i = g(\sigma_{ij}\gamma_j)g^{-1}$$

Given $\sigma \in SO(4)$, can we determine $g \in SU(4)$?
At a pt p, yes! b/c wrt each trivialization, $c(\tilde{e}_i)$ is a Clifford action on $\mathbb{C}^4$ so both isomorphic to Δ_p at p

The TROUBLE is that g is <u>not</u> uniquely determined.
If g works $\Rightarrow$ so does $-g$ (e.g)
So choice of g at a pt might not be possible to be globally consistent.

When M has spin structure, we can construct Δ. *Translates to some kinds of space times can't have spinors.
Enrique surface, $\mathbb{C}P^2$,
1st chern class has torsion.

*Not being able to construct Δ as a bundle over orientable M

Regardless, locally every Clifford bundle is $V \otimes \Delta$ for some bundle V

Recall from last time, we define the twisting curvature for a Clifford bundle.

S - Clifford bundle

Riemann endomorphism:

$$R^S(X,Y) = \frac{1}{4}\sum_{i,j} c(e_i)c(e_j)\langle R(X,Y)e_i, e_j\rangle$$

Riemann curv tensor.

Then, if $K(X,Y)$ is curvature of ∇^S, we can write

$$K(X,Y) = R^S(X,Y) + \underbrace{F^S(X,Y)}_{\text{twisting curvature}}$$

s.t. $F^S(X,Y)$ commutes w/ Clifford mult.

If E is a vector bundle (not Clifford bundle) & M is spin, we can construct a Clifford bundle S

$$S = E \otimes \Delta$$

Choices of Δ
$\exists$ sometimes diff choices of Δ. So for now assume we chose one.

We get that

Then,

$$\nabla^S = \nabla^E \otimes 1 + 1 \otimes \nabla^\Delta$$

$$K^S = F^E \otimes 1 + 1 \otimes K^\Delta$$

$$F^S = F^E \otimes 1$$

~~Given a Clifford bundle~~

Call a $\mathbb{C}$-linear endomorphism of a Clifford module is a Clifford endomorphism if it commutes w/ the Clifford action.

Note the only Clifford endomorphism of Δ are complex multiples of the identity, $\mathbb{C}I$.

In general, on any Clifford module, ~~we can exp~~

$$W = V \otimes \Delta$$

Clifford endomorphisms are then just

$$\Phi \otimes 1 \quad \text{for } \Phi \in \mathrm{End}(V)$$

Then we can define the relative trace of a Clifford endomorphism

$\Phi \otimes 1$, (locally at least)

$$tr^{W/\Delta}(\Phi \otimes 1) = tr(\Phi)$$

Now, given a Clifford bundle S, we define the relative chern character

$$ch(S/\Delta) = tr^{S/\Delta}(\exp(-F^S/2\pi i))$$

This works for any Clifford bundle even when Δ is not defined globally.
If Δ not defined globally, S/Δ is just notation.
& we have $ch(S/\Delta)$ defined locally & is well defined globally

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If M is spin, and we take $S = E \otimes \Delta$, then

$$ch(S/\Delta) = ch(E).$$

(b/c then $F^S = F^E \otimes 1$.)

We have $\Delta = \Delta^+ \oplus \Delta^-$

$$c(e_1) = \left(\begin{array}{cc|cc} & & -1 & \\ & & & 1 \\ \hline 1 & & & \\ & -1 & & \end{array}\right)$$

$$\Delta = \mathbb{C}^4 \text{ w/ this action}$$
$$= \mathbb{C}^2 \oplus \mathbb{C}^2 = \Delta^+ \oplus \Delta^-$$

Clifford action takes $\Delta^+ \longrightarrow \Delta^-$
$\Delta^- \longrightarrow \Delta^+$

Individually they are not Cliffmodules

$E \otimes \Delta$

$$(E \otimes \Delta)^+ = E \otimes \Delta^+$$
$$(E \otimes \Delta)^- = E \otimes \Delta^-$$

We can define this grading more generally (in dim 4)

Let $e_1, \dots, e_4$ be a local orthonormal frame

We define the volume element

$$w = c(e_1)c(e_2)c(e_3)c(e_4)$$

$$w^2 = 1$$

So, eigenvalues of ω are ± 1

Let $S^{\pm}$ be ± 1 eigenspace

Clifford action

$$c(e_i)\omega = -\omega c(e_i)$$

$\Rightarrow$ Clifford action maps

$$S^+ \longrightarrow S^-$$
$$S^- \longrightarrow S^+$$

CLAIM

$$D : \Gamma(S^{\pm}) \longrightarrow \Gamma(S^{\mp})$$

Pf Check at center of nice orthonormal frame

(i.e) $(\nabla_{e_i} e_j)_p = 0$)

Say $s \in \Gamma(S^+)$

$$(\omega D s)_p = (c(e_1)\cdots c(e_4)\sum_i c(e_i)\nabla_i s)_p$$
$$= -(c(e_i)c(e_1)\cdots c(e_4)\nabla_i s)_p$$

So $(\omega D s)_p = (\sum_i - c(e_i)\nabla_i c(e_1)\cdots c(e_4)s)_p$

$$= -\sum_i c(e_i)\nabla_i s$$
$$= -Ds$$

□

So we define $D = D_+ + D_-$ $\quad D_+ : \Gamma(S^+) \rightarrow \Gamma(S^-)$

$$D_+ = D|_{S^+} \qquad D_- = D|_{S^-}$$

Last time we showed $D^* = D$ $\quad \langle D s_1, s_2\rangle = \langle s_1, D^* s_2\rangle$

$$\Rightarrow \quad D_+^* = D_- \qquad D_-^* = D_+$$

We define the index of D

$$\operatorname{ind}(D) = \dim(\ker(D_+)) - \dim(\ker(D_-))$$
$$= \dim(\ker(D_+)) - \dim(\ker(D_+^*))$$

INDEX THEOREM

Let M be compact, even dimensional & oriented.
Let S be a Clifford bundle on M w/ associated Dirac operator D. Then

$$\operatorname{ind}(D) = \int_M \hat{A}(TM) \wedge ch(S/\Delta)$$

We've really only looked at Δ for $\dim = 4$.
Can get index thms for more general elliptic operators but this is the best one.

On the RHS messy, on LHS it's an integer.

At a very basic level, index thm is

$$\operatorname{ind}(D) = \int \text{something involving curvature}$$

Gauss Bonnet simplest ex.

If M is spin

$$S = E \otimes \Delta$$

Then $\operatorname{ind}(D) = \int_M \hat{A}(TM) \wedge ch(E)$

Take $S = \Delta$

$$\operatorname{ind}(D) = \int_M \hat{A}(TM) \wedge 1 = \int_M \hat{A}(TM)$$

So if M spin, $\dim(M) = 4$ & $S = \Delta$

$$\int_M \frac{1}{24} p_1(TM) = \operatorname{ind}(D) \in \mathbb{Z}$$

So $p_1(M)$ is divisible by 24.
They noticed this fact before they knew (showed) the index thm.

Gauss Bonnet: what is the Dirac operator?
easier w/ another grading $\quad M =$ surface, orientable

$\Lambda^\bullet T^*M$ Cliffor bundle

$D = d + d^*$

grading s.t. $\quad \Lambda^\bullet T^*M = (\Lambda^0 T^*M \oplus \Lambda^2 T^*M) \oplus (\Lambda^1 T^*M)$

$$\mathrm{ind}(d+d^*) = \dim(\ker(d+d^*|_{\Lambda^{even}})) - \dim(\ker(d+d^*|_{\Lambda^{odd}}))$$

$$\dim(\ker(d+d^*|_{\Lambda^p T^*M})) \simeq H^p(M).$$

Hodge thm
can always find harmonic rep of every cohomology class.

$$\Rightarrow \mathrm{ind}(d+d^*) = \dim(H^0 \oplus H^2) - \dim(H^1)$$
$$= 2 - 2g$$

Other side takes a little more effort.