Math: PART III: Clifford actions on vector bundles

DEFINITION: Let $S \to M$ be a complex vector bundle, A is a Clifford action on S if A is a (smoothly varying) map (in x)

$$c_x : T_x M \to \mathrm{End}(S_x)$$

such that

$$c_x(V_1)\, c_x(V_2) + c_x(V_2)\, c_x(V_1) = -2 \langle V_1, V_2 \rangle \, \mathrm{Id}$$

where $\langle , \rangle$ from Riemannian metric on M

We will say S is a Clifford bundle (nonstd name) if S has a Clifford action satisfying

* the Clifford action is skew adjoint

$$\langle c(V) s_1, s_2 \rangle = - \langle s_1, c(V) s_2 \rangle$$

$s_1, s_2 \in S_x$, $V \in T_x M$

(S has a hermitian metric & connection)

* the Clifford connection is compatible with the connection on S_1

Y, X is some v.f. $f \in \Gamma(S)$

$$\nabla^S_X \left(c(Y) f \right) = c(\nabla_X Y) f + c(Y) \nabla^S_X f$$

∇^S is the connection on S

∇ is the levi civita connection on M.

Equivalently $c(Y) \in \Gamma(\mathrm{End}(S))$ & $\nabla_X c(Y) = c(\nabla_X Y)$

EXAMPLES

On an oriented Riemannian manifold M, we use the metric g to identify T_x^*M & $T_x M$.

$$X_p^b(Y) = g(X_p, Y_p) \quad \text{lowering the index}$$

We will let $\mathcal{E}(X)$ be wedge with X^b

$$\mathcal{E}(X)\alpha = X^b \wedge \alpha$$

Let $i(X)$ be the interior product w/ X

$$i(X): \Omega^p(M) \to \Omega^{p-1}(M)$$

$$(i_X(\alpha))(Y_1, \ldots, Y_{p-1}) = \alpha(X, Y_1, \ldots, Y_{p-1})$$

hook or contraction of α w/ X.

We define a Clifford action on $\Lambda^\bullet T^*M$ by

$$c(X)\alpha = \mathcal{E}(X) \wedge \alpha - i(X)\alpha$$

This makes $\Lambda^\bullet T^*M$ into a Clifford bundle

For $\{e_1, \ldots, e_n\}$ an orthonormal basis of $T_x M$,

$$c(e_i)c(e_j) + c(e_j)c(e_i) = -2\langle e_i, e_j\rangle$$
$$= -2\delta_{ij}$$

$$\Rightarrow c(e_i)c(e_j) = \begin{cases} -1 & i=j \\ -c(e_j)c(e_i) & i \neq j \end{cases}$$

To check that $\Lambda^\bullet T^*M$ is a Clifford bundle, Pick an orthonormal frame.

useful:

$$i(X)(\alpha \wedge \beta) = (i(X)\alpha) \wedge \beta + (-1)^{\deg|\alpha|} \alpha \wedge (i_X \beta)$$

Also useful:

- $\nabla_Y(\mathcal{E}(X)\alpha) = \mathcal{E}(\nabla_Y X)\alpha + \mathcal{E}(X)\nabla_Y \alpha$

- $i(X) = \mathcal{E}(X)^*$ from these we can find that it's a Clifford action useful exercise

EXAMPLE:

Let $M = \mathbb{R}^4$, define a Clifford action on the trivial $\mathbb{C}^4$ bundle over $\mathbb{R}^4$ by

$$c(e_1) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad c(e_2) = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}$$

$$c(e_3) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{bmatrix} \qquad c(e_4) = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

connections $= 0$

skew adjoint b/c all these matrices are skew hermitian

all of them square to -1 & anticommute

Extend linearly to any element of the tangent space

This is the spin representation in dim 4 (of Clifford algebra)

NOTE: ~~One thing~~

We can split $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$

$\qquad\qquad\qquad\qquad S^+ \qquad S^-$

$c(X): S^+ \longrightarrow S^-$

$\qquad\quad S^- \longrightarrow S^+$

In even dim the Clifford algebra has a unique irreducible representation. Also gives a representation of spin group but is not irreducible for spin group

$\mathrm{Spin}_4 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$
↑ universal cover of $SO(n)$

DEFn: The Dirac operator on S (S is still a Clifford bundle) is the 1st order differential operator given locally by

$$DS = \sum_i c(e_i)\nabla_{e_i} S$$

where e_i is an orthonormal frame

This is in fact well defined

We can also define the Dirac operator more invariantly

$$\begin{tikzcd} \Gamma(S) \arrow[r, "\nabla"] \arrow[ddr, "D"'] & \Gamma(T^*M \otimes S) \arrow[d, "\#"] \\ & \Gamma(TM \otimes S) \arrow[d] \\ & \Gamma(S) \end{tikzcd}$$

$$\begin{tikzcd} (X \otimes S) \arrow[d, mapsto] \\ c(X)S \end{tikzcd}$$

Given a fibre metric, recall we get an L^2 inner product on $\Gamma(S)$

$$\langle\langle s_1, s_2 \rangle\rangle = \int_M \langle s_1, s_2 \rangle \, d\mathrm{vol}_M$$

PROPOSITION: D is self adjoint $\qquad \nabla_i = \nabla_{e_i}$

Proof: Integration by parts tells us that $\nabla_i^* = -\nabla_i$ (shaky)

$$D^* = \sum (c(e_i)\ \nabla_{e_i})^*$$

$$= \sum + \nabla_{e_i}^* c(e_i)^* = \sum \nabla_{e_i} c(e_i)$$

Need only check $(Ds)(p) = (D^*s)(p)$

So check in a nice orthonormal frame, nice being

$$(\nabla_{e_i} e_j)_p = 0$$

$$\Rightarrow D^* = \sum_i \nabla_{e_i} c(e_i) = \underbrace{c(\nabla_{e_i} e_i)}_{0 \text{ at } p} + c(e_i)\nabla_{e_i}$$

$$= D$$

□

EXAMPLE: Let $S = \Lambda^\bullet TM$

Then

$$D = d + d^*$$

"*1/2 proof*" Work in Riemann normal coordinates, then

$$d(\alpha_I \, dx^I) = \frac{\partial \alpha_I}{\partial x^j} dx^j \wedge dx^I$$

$$= \varepsilon(dx^j)\left(\frac{\partial \alpha_I}{\partial x^j} dx^I\right)$$

$$= \varepsilon(dx^j)(\nabla_{e_i}\alpha)_p$$

$$d = \sum_i \varepsilon(e_i)\nabla_{e_i}$$

$$\Rightarrow d^* = -\sum_i i(e_i)\nabla_{e_i}$$

$c(e_i)$ clifford action!

$$\Rightarrow D = \sum_i \big(\varepsilon(e_i) - i(e_i)\big)\nabla_{e_i}$$

$\square$

In particular

$$D^2 = (d + d^*)(d + d^*)$$

$$= d^*d + dd^*$$

$$= \Delta \quad \text{the Hodge Laplacian}$$

when restricted to functions, its the usual laplacian

Can think of this as motivation to define the dirac operator as a square root of laplacian

$$d^* = \pm * d *$$

In general, we have

PROPOSITION: (Weitzenbock or Lichnerowicz formula

$$D^2 s = \nabla^* \nabla s + \hat{K} s$$

where

$$\hat{K} = \sum_{i<j} c(e_i)c(e_j) K_{ij}$$

curvature of connection on S

$$\nabla : \Gamma(S) \longrightarrow \Gamma(T^*M \otimes S)$$

Proof: Work at centre of Riemann normal coords

$$D^2 s = \sum_{i,j} c(e_i)\nabla_i c(e_j)\nabla_j s$$

$$= \sum_{i,j} c(e_i)c(e_j)\nabla_i \nabla_j s$$

$$= \sum_{j=i} c(e_i)c(e_i)\nabla_i \nabla_i s + \sum_{i<j} c(e_i)c(e_j)\nabla_i \nabla_j s$$

$$+ \sum_{i>j} c(e_i)c(e_j)\nabla_i \nabla_j s$$

$$= \underbrace{\sum_i (-\nabla_i \nabla_i s)}_{\nabla^*\nabla s \text{ term}} + \underbrace{\sum_{i<j} c(e_i)c(e_j) \underbrace{[\nabla_i\nabla_j - \nabla_j\nabla_i]}_{= K_{ij} \text{ b/c we are in a good patch}} s}_{\hat{K}s \text{ term.}}$$

□

Just as a Corollary: (Bochner-Formula)

For $S \in \Lambda^\bullet TM$ w/ $D = d + d^*$

$$\Delta s = \nabla^* \nabla s + \hat{K} s$$

↳ just involves Riemann curvature

THM (Bochner) If the least eigenvalue of $\hat{K}$ at each point of M is strictly positive, then there are no nonzero solutions of $D^2 s = 0$

Proof: $\langle \hat{K}s, s\rangle \geq c\|s\|$ for some $c > 0$

If $D^2 s = 0 \Rightarrow 0 = \langle D^2 s, s\rangle = \langle \nabla^*\nabla s + \hat{K}s, s\rangle$

$$= \|\nabla s\|^2 + \langle \hat{K}s, s\rangle$$

$$\geq \|\nabla s\|^2 + c\|s\|^2 \geq 0.$$

$$\Rightarrow s = 0$$

In practice, one way to construct Clifford bundles is by taking a vector bundle $E \to M$, and $\Delta \to M$ locally given by the spin representation, take $S = E \otimes \Delta$ clifford action is then clif action on Δ

The curvature of $E \otimes \Delta$ is of the form

$$K = \underbrace{F^E \otimes 1}_{\substack{\text{interested}\\ \text{in studying}\\ \text{this part}}} + \underbrace{1 \otimes K^\Delta}_{\text{fixed}}$$

Note: As an endomorphism of S, $F^E \otimes 1$ commutes with $c(X)$ (which only acts on Δ part)

In fact:

LEMMA: As endomorphisms of S

$$[\underset{\substack{\uparrow\\ \text{curvature of } S}}{K(X,Y)}, c(Z)] = c(R(X,Y)Z)$$

<u>Proof</u>: $\nabla_X \nabla_Y c(Z) = \nabla_X \big(c(\nabla_Y Z) + c(Z)\nabla_Y\big)$
compatibility w/ connection

$$= c(\nabla_X \nabla_Y Z) + c(\nabla_Y Z)\nabla_X + c(\nabla_X Z)\nabla_Y + c(Z)\nabla_X \nabla_Y$$

$\Rightarrow$

$$[\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}, c(Z)]$$

$\square$

~~In general~~ DEFN: We define the Riemann endomorphism $\mathcal{R}^S$ of S to be

$$\mathcal{R}^S(X,Y) = \frac{1}{4} \sum_{k,l} c(e_k)c(e_l) \langle R(X,Y)e_k, e_l \rangle$$

LEMMA: As endomorphisms of S,

$$[R^S(X,Y), c(Z)] = c(R(X,Y)Z)$$

Note

$$[K(X,Y), c(Z)] = c(R(X,Y)Z) = [R^S(X,Y), c(Z)]$$

So, **COROLLARY:**

$$[K(X,Y) - R^S(X,Y), c(Z)] = 0$$

So $K(X,Y) - R^S(X,Y)$ commutes w/ the Clifford action.

We define $F^S := K(X,Y) - R^S(X,Y)$ and call it the <u>twisting curvature</u> of S

F^S plays the role of curvature in $E \otimes \Delta$ & can compute F^E from F^S ($R^S(X,Y)$ forced on us from compatibility w/ Clifford action)

PROPOSITION:

$$D^2 s = \nabla^* \nabla s + \hat{F} s + \frac{1}{4} \mathcal{K} s$$

$\mathcal{K}$ ← scalar curvature of M

$$\hat{F} = \sum c(e_i) c(e_j) F^S_{ij}$$