

# Matt: PART III: Clifford actions on vector bundles

DEFINITION: Let  $S \rightarrow M$  be a <sup>complex</sup> vector bundle,  $A$  is a Clifford action on  $S$  if  $A$  is a (smoothly varying) map (in  $x$ )

$$C_x: T_x M \rightarrow \text{End}(S_x)$$

such that

$$C_x(v_1) C_x(v_2) + C_x(v_2) C_x(v_1) = -2 \langle v_1, v_2 \rangle \text{Id}$$

where  $\langle, \rangle$  from Riemannian metric on  $M$

We will say  $S$  is a Clifford bundle (nonstol name) if  $S$  has a Clifford action satisfying

\* the Clifford action is skew adjoint

$$\langle c(v) s_1, s_2 \rangle = - \langle s_1, c(v) s_2 \rangle$$

$$s_1, s_2 \in S_x, v \in T_x M$$

( $S$  has a hermitian metric & connection)

\* the Clifford connection is compatible with the connection on  $S$

$Y, X$  is some v.f.  $f \in \Gamma(S)$

$$\nabla_x^S (c(Y) f) = c(\nabla_x Y) f + c(Y) \nabla_x^S f$$

$\nabla^S$  is the connection on  $S$

$\nabla$  is the Levi-Civita connection on  $M$ .

Equivalently  $c(Y) \in \Gamma(\text{End}(S)) \quad \& \quad \nabla_x c(Y) = c(\nabla_x Y)$

### EXAMPLES

On an oriented Riemannian manifold  $M$ , we use the metric  $g$  to identify  $T_x^*M \cong T_x M$ .

$$X_p^\flat(Y) = g(X_p, Y_p) \quad \text{lowering the index}$$

We will let  $\mathcal{E}(X)$  be wedge with  $X^\flat$

$$\mathcal{E}(X)\alpha = X^\flat \wedge \alpha$$

Let  $i(X)$  be the interior product w/  $X$

$$i(X): \Omega^p(M) \rightarrow \Omega^{p-1}(M)$$

$$(i_X(\alpha))(Y_1, \dots, Y_{p-1}) = \alpha(X, Y_1, \dots, Y_{p-1})$$

hook or contraction of  $\alpha$  w/  $X$ .

We define a Clifford action on  $\Lambda^* T^*M$  by

$$c(X)\alpha = \mathcal{E}(X)\alpha - i(X)\alpha$$

This makes  $\Lambda^* T^*M$  into a Clifford bundle

For  $\{e_1, \dots, e_n\}$  an orthonormal basis of  $T_x M$ ,

$$c(e_i)c(e_j) + c(e_j)c(e_i) = -2\langle e_i, e_j \rangle = -2\delta_{ij}$$

$$\Rightarrow c(e_i)c(e_j) = \begin{cases} -1 & i=j \\ -c(e_j)c(e_i) & i \neq j \end{cases}$$

To check that  $\Lambda^* T^*M$  is a Clifford bundle, ~~we~~

Pick an orthonormal frame.

useful:

$$i(X)(\alpha \wedge \beta) = (i(X)\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge (i_X \beta)$$

Also useful:

- $\nabla_y (\varepsilon(X) \alpha) = \varepsilon(\nabla_y X) \alpha + \varepsilon(X) \nabla_y \alpha$
- $i(X) = \varepsilon(X)^*$  from these we can find that it's a Clifford action  
useful exercise

EXAMPLE:

Let  $M = \mathbb{R}^4$ , define a Clifford action on the trivial  $\mathbb{C}^4$  bundle over  $\mathbb{R}^4$  by

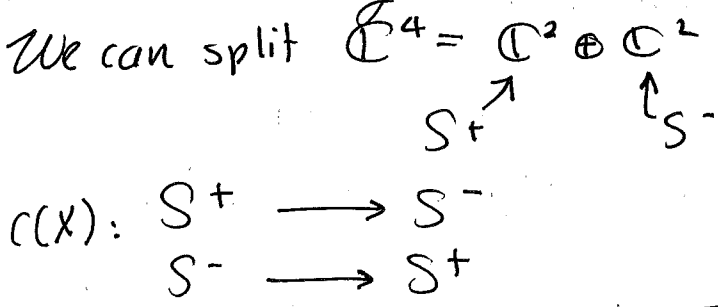
$$c(e_1) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad c(e_2) = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & i & 0 \end{bmatrix}$$

$$c(e_3) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ +1 & 0 & 0 \end{bmatrix} \quad c(e_4) = \begin{bmatrix} 0 & 0 & i \\ 0 & -i & 0 \\ i & 0 & 0 \end{bmatrix}$$

connections = 0  
 skew adjoint b/c all these matrices are skew hermitian  
 all of them square to -1 & anticommute  
 Extend linearly to any element of the tangent space

This is the spin representation in dim 4 (of Clifford algebra)

NOTE: ~~connection~~



In even dim the Clifford algebra has a unique irreducible representation. Also gives a representation of spin group but is not irreducible for spin group

$\text{Spin}_4 = \text{SU}_2 \otimes \text{SU}_2$   
 ↑ universal cover of  $\text{SO}(4)$

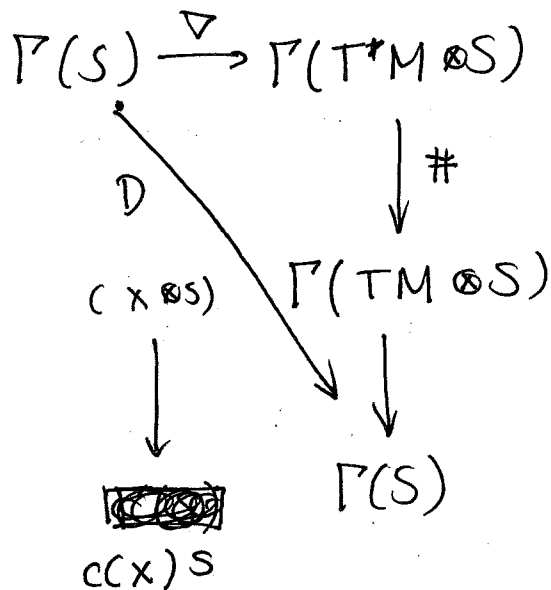
DEFN: The Dirac operator on  $S$  ( $S$  is still a Clifford bundle) is the 1st order differential operator given locally by

$$D_S = \sum_i c(e_i) \nabla_{e_i} S$$

where  $e_i$  is an orthonormal frame

This is in fact well defined

We can also define the Dirac operator more invariantly



Given a fibre metric, recall we get an  $L^2$  inner product on  $\Gamma(S)$

$$\langle\langle S_1, S_2 \rangle\rangle = \int_M \langle S_1, S_2 \rangle d\text{vol}_M$$

PROPOSITION:  $D$  is self adjoint

Proof: Integration by parts tells us that  $\nabla_i^* = -\nabla_i$

$$\nabla_i = \nabla_{e_i}$$

(shaky) ~~□~~

$$D^* = \sum_i (c(e_i) \nabla_{e_i})^*$$

$$= \sum_i \nabla_{e_i}^* c(e_i) = \sum_i \nabla_{e_i} c(e_i)$$

Need only check  $(D_S)(p) = (D^*S)(p)$

So check in a nice orthonormal frame, nice being

$$(\nabla_{e_i} e_j)_p = 0$$

$$\Rightarrow D^* = \sum_i \nabla_{e_i} c(e_i) = c(\nabla_{e_i} e_i) + c(e_i) \nabla_{e_i}$$

$$= D$$

□

EXAMPLE: Let  $S = \Lambda^0 TM$

Then

$$D = d + d^*$$

"1/2 proof" Work in Riemann normal coordinates, then

$$\begin{aligned} d(\alpha_I dx^I) &= \frac{\partial \alpha_I}{\partial x^J} dx^J \wedge dx^I \\ &= \varepsilon(dx^j) \left( \frac{\partial \alpha_I}{\partial x^j} dx^I \right) \\ &= \varepsilon(dx^j) (\nabla_{e_j} \alpha)_I \end{aligned}$$

$$d = \sum_i \varepsilon(e_i) \nabla_{e_i}$$

$$\Rightarrow d^* = -\sum_i i(e_i) \nabla_{e_i}$$

$$\Rightarrow D = \sum_i (\varepsilon(e_i) - i(e_i)) \nabla_{e_i} \quad \square$$

$c(e_i)$   
clifford action!

In particular

$$\begin{aligned} D^2 &= (d + d^*)(d + d^*) \\ &= d^*d + dd^* \\ &= \Delta \quad \text{the Hodge Laplacian} \end{aligned}$$

when restricted to functions, it's the usual laplacian

Can think of this as motivation to define the dirac operator as a square root of laplacian

$$d^* = \pm * d *$$

~~Prop~~  
In general, we have  
PROPOSITION:  
(Weitzenbock or Lichnerowicz formula)

$$D^2 S = \nabla^* \nabla S + \hat{K} S$$

where

$$\hat{K} = \sum_{i < j} c(e_i) c(e_j) K_{ij}$$

↑  
curvature of connection on  $S$

$$\nabla: \Gamma(S) \rightarrow \Gamma(T^*M \otimes S)$$

Proof: Work at centre of Riemann normal coords

$$D^2 S = \sum_{i,j} \square c(e_i) \nabla_i c(e_j) \nabla_j S$$

$$= \sum_{i,j} c(e_i) c(e_j) \nabla_i \nabla_j S$$

$$= \sum_{j=i} c(e_i) c(e_i) \nabla_i \nabla_i S + \sum_{i < j} c(e_i) c(e_j) \nabla_i \nabla_j S$$

$$+ \sum_{i > j} c(e_i) c(e_j) \nabla_i \nabla_j S$$

$$= \underbrace{\sum_i (-\nabla_i \nabla_i s)}_{\nabla^* \nabla s \text{ term}} + \underbrace{\sum_{i < j} c(e_i)(e_j) [\nabla_i \nabla_j - \nabla_j \nabla_i] s}_{\hat{K} s \text{ term.}}$$

=  $K_{ij}$  bc we are in a good patch

□

Just as a Corollary: (Bochner-Formula)

For  $S \in \Lambda^0 TM$  w/  $D = d + d^*$

$$\Delta S = \nabla^* \nabla S + \hat{K} S$$

↑ just involves Riemann curvature

THM (Bochner) If the least eigenvalue of  $\hat{K}$  at each point of  $M$  is strictly positive, then there are no nonzero solutions of  $D^2 S = 0$

Proof:  $\langle \hat{K} S, S \rangle \geq c \|S\|^2$  for some  $c > 0$

$$\begin{aligned} \text{If } D^2 S = 0 &\Rightarrow 0 = \langle D^2 S, S \rangle = \langle \nabla^* \nabla S + \hat{K} S, S \rangle \\ &= \|\nabla S\|^2 + \langle \hat{K} S, S \rangle \\ &\geq \|\nabla S\|^2 + c \|S\|^2 \geq 0. \end{aligned}$$

~~0 = 0~~

$$\Rightarrow S = 0$$

In practice, one way to construct Clifford bundles is by taking a vector bundle  $E \rightarrow M$ , and  $\Delta \rightarrow M$  locally given by the spin representation, take  $S = E \otimes \Delta$  clifford action is then cliff action on  $\Delta$

The curvature of  $E \otimes \Delta \square$  of the form

$$K = \underbrace{F^E \otimes 1}_{\text{interested in studying this part}} + \underbrace{1 \otimes K^\Delta}_{\text{fixed}}$$

Note: As an endomorphism of  $S$ ,  $F^E \otimes 1$  commutes with  $c(X)$  (which only acts on  $\Delta$  part)

In fact:

LEMMA: As endomorphisms of  $S$

$$[K(X, Y), c(Z)] = c(R(X, Y)Z)$$

↑  
curvature of  $S$

Proof:  $\nabla_x \nabla_y c(Z) = \nabla_x (c(\nabla_y Z) + c(Z) \nabla_y)$   
 compatibility w/ connection

$$= c(\nabla_x \nabla_y Z) + c(\nabla_y Z) \nabla_x + c(\nabla_x Z) \nabla_y + c(Z) \nabla_x \nabla_y$$

$\Rightarrow$   ~~$\nabla_x \nabla_y c(Z)$~~   
 ~~$c(\nabla_x \nabla_y Z) + c(\nabla_y Z) \nabla_x + c(\nabla_x Z) \nabla_y + c(Z) \nabla_x \nabla_y$~~

$$[\nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[X, Y]}, c(Z)] = 0 \quad \square$$

~~general~~ DEFIN: We define the Riemann endomorphism  $R^S$  of  $S$  to be

$$R^S(X, Y) = \frac{1}{4} \sum_{k, l} c(e_k) c(e_l) \langle R(X, Y) e_k, e_l \rangle$$

LEMMA: As endomorphisms of  $S$ ,

$$[R^S(X, Y), c(Z)] = c(R(X, Y)Z)$$

Note

$$\begin{aligned} [K(X, Y), c(Z)] &= c(R(X, Y)Z) \\ &= [R^S(X, Y), c(Z)] \end{aligned}$$

So, COROLLARY:

$$[K(X, Y) - R^S(X, Y), c(Z)] = 0$$

So  $K(X, Y) - R^S(X, Y)$  commutes w/ the Clifford action.

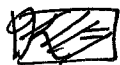
We define  $F^S := K(X, Y) - R^S(X, Y)$  and call it the twisting curvature of  $S$

$F^S$  plays the role of curvature in  $E \otimes \Delta$  & can compute  $F^E$  from  $F^S$  ( $R^S(X, Y)$  fixed on  $\psi$  from compatibility w/ Clifford action)

PROPOSITION:

$$D^2 S = \nabla^* \nabla S + \hat{F} S + \frac{1}{4} \mathcal{K} S$$

↑ scalar curvature of  $M$



$$\hat{F} = \sum c(e_i)c(e_j) F_{ij}^S$$