Characteristic classes part II

Recap $E \longrightarrow M$ with connection ∇, and structure group $U(n)$, $SU(n)$ (so we have a hermitian metric & ∇ is compatible with it)

Locally $\nabla = d + A$
matrix valued 1-form

Curvature $F = dA + A \wedge A$
$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$

Last time we considered invariant polynomials
$$p : \mathfrak{gl}_m(\mathbb{C}) \longrightarrow \mathbb{C}$$
invariant meaning that $p(gXg^{-1}) = p(X)$ for $g \in GL_m(\mathbb{C})$

Under gauge transformation
$$A \longrightarrow g\,d\,g^{-1} + gAg^{-1}$$
$$F \longrightarrow gFg^{-1}$$

One example of an invariant polynomial was
$$c_k(X) = \mathrm{tr}(\Lambda^k X) \qquad k \leq m.$$
$\Lambda^k X$ acts on $\Lambda^k \mathbb{C}^m$

Actually, these polynomials can be thought of as symmetric polynomials in the eigenvalues

$\mathrm{tr}(\Lambda^k X)$ are elementary symmetric polynomials

NOTE: $\det(1+qX) = \sum q^k c_k(X)$
nice to compute things.

Define $c_k(E) := c_k\left((+2\pi i)^{-1} F_\nabla\right)$ kth Chern class of E

curvature is a matrix valued 2-form, so we plug into polynomials, two forms commute so algebra works out nicely & $c_k(E)$ is a $2k$-form.

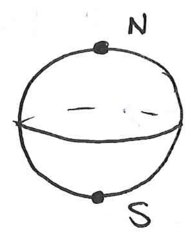
<u>Last time</u>:

$[c_k(E,\nabla)] \in H^{2k}_{dR}(M)$ is independent of choice of ∇

$d c_k(E) = 0$ and picking a diff ∇'

$c_k(E,\nabla) = c_k(E,\nabla') + d\eta$

EXAMPLE
$M = S^2$. We have coordinate charts given by a stereographic projection.



(x^1, x^2) away from north pole, N
(y^1, y^2) away from south pole, S

Let $z = x^1 + i x^2$
$w = y^1 - i y^2$ ← Charts oriented diff.

Change of coods is $z = \dfrac{1}{w}$

we are identifying $S^2 \simeq \mathbb{CP}^1$

Define a $U(1)$ bundle by transition functions

$$g(z) = \frac{z^k}{|z|^k} \in U(1) \text{ for some } k \in \mathbb{Z}$$

$$A \longrightarrow g\,dg^{-1} + gAg^{-1}$$

when $z \to w$

Pick connection form on z-patch
$$A = \frac{k\,\bar{z}dz - zd\bar{z}}{2(1+|z|^2)} \in i\mathbb{R} = U(1)$$

← same k

Can't pick just any connection, need connection defined on z patch everywhere away from N but that when $z \to w$ we have it defined on N.

$$A \longrightarrow g\,dg^{-1} + gAg^{-1}$$

$$= \frac{z^k}{|z|^k}\, d\, \frac{z^{-k}}{|z|^{-k}} + \frac{k\,\bar{z}dz - zd\bar{z}}{2(1+|z|^2)}$$

$$= \frac{k\, zd\bar{z} - \bar{z}dz}{2|z|^2(1+|z|^2)}$$ well defined on w-patch not on z-patch!

$$= k\,\frac{\frac{1}{w}d\left(\frac{1}{\bar{w}}\right) - \frac{1}{\bar{w}}d\left(\frac{1}{w}\right)}{2\left|\frac{1}{w}\right|^2\left(1+\frac{1}{|w|^2}\right)}$$

$$= k\,\frac{\bar{w}dw - wd\bar{w}}{2(1+|w|^2)}$$ looks the same at N & S

Seems like you can't pick just any connection, $\exists$ some topological restriction to define "nicely" over each chart.

$$F_\nabla = dA + \underbrace{A \wedge A}_{=0 \text{ in } U(1)}$$

$$= \frac{-k\,dz \wedge d\bar{z}}{(1+|z|^2)^2}$$

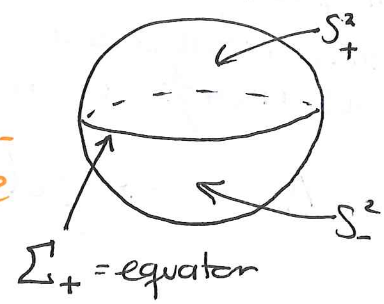
$$c_1(E) = \mathrm{tr}\left(-\frac{1}{2\pi i} F_\nabla\right) = \frac{k}{2\pi i} \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} = -\frac{k}{4\pi} \mathrm{Vol}_{S^2}$$

$$\int_{S^2} c_1(E) = -k$$

Note

$$\int_{S^2} -\frac{1}{2\pi i} F_\nabla = \int_{S^2_+} -\frac{1}{2\pi i} \underbrace{F_\nabla}_{= dA_+} + \int_{S^2_-} -\frac{1}{2\pi i} \underbrace{F_\nabla}_{= dA_-}$$

b/c we have $U(1)$ bundle



$$\underset{\text{(Stokes)}}{=} \int_\Sigma \left(-\frac{1}{2\pi i} A_+\right) - \int_\Sigma \left(-\frac{1}{2\pi i} A_-\right)$$

But $A_- = g\,dg^{-1} + gAg^{-1} = g\,dg^{-1} + A_+$ (b/c $U(1)$)

$$\Rightarrow \int_{S^2} -\frac{1}{2\pi i} F_\nabla = \frac{-1}{2\pi i}\left[\int_\Sigma A_+ - \left(\int_\Sigma g\,dg^{-1} + A_+\right)\right]$$

$$= \frac{1}{2\pi i} \int_\Sigma g\,dg^{-1}$$

Chern class determined by how charts transition/match up at the equator

Define the total chern class to be

$$c(E) = 1 + c_1(E) + c_2(E) + \dots + c_m(E)$$

$$c(E) = \det\left(1 - \frac{1}{2\pi i} F_\nabla\right) \in H^\bullet_{dR}(M) = \bigoplus_i H^i_{dR}(M)$$

Properties of the Chern class:

★ $c(E_1 \oplus E_2) = c(E_1) \wedge c(E_2)$

we can see this easily b/c

$$F^{E^1 \oplus E^2} = F^{E^1} \oplus F^{E^2}$$

So, ~~det~~ $\begin{bmatrix} F^{E_1} & 0 \\ 0 & F^{E_2} \end{bmatrix}$

$\Rightarrow c_1(E_1 \oplus E_2) = c_1(E_1) + c_1(E_2)$

$c_2(E_1 \oplus E_2) = c_2(E_1) + c_2(E_2) + c_1(E_1) \wedge c_1(E_2)$

★ For $f: N \to M$ we can define the pullback bundle f^*E on N. Then

$$c(f^*E) = f^*c(E)$$

Easy to see by pulling back connection, $\Rightarrow$ getting pullback of curvature.

★ $c_k(E)$ is an integral element of $H^{2k}(M)$

If you take some element of $[c_k(E)]$ & integrate (over M) we get an integer. Always happens for compact things.

★ $c_k(E) = 0$ is trivial if E is trivial

(as elements of de Rham cohomology, not considering torsion)

These characterize $c_k(E)$ + some normalization properties

We can also consider invariant formal power series on $\mathfrak{gl}_m(\mathbb{C})$

$P(X)$

Any 2-form is nilpotent in $H_{dR}^{\bullet}(M)$, so $P(F)$ always converges (& is a polynomial)

EXAMPLE: The Chern character is defined to be

$$ch(E) = tr(\exp(-\frac{i}{2\pi} F_\nabla))$$

Which has the following properties:

$$ch(E_1 \otimes E_2) = ch(E_1)\,ch(E_2)$$
$$ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$$

So ch is a "ring" homomorphism

Given f holomorphic around 0 and $f(0) = 1$.
(so we have a power series!)

$$\Pi_f(X) = \det(f(-\frac{1}{2\pi i} X))$$ and the

associated characteristic class

$$\det(f(-\frac{1}{2\pi i} F_\nabla))$$ is called the Chern f-genus

The total Chern class is the Chern f-genus associated to

$$f(z) = 1 + z$$

Thinking of Π_f as a polynomial of eigenvalues, we get that $\Pi_f(X) = \prod_{j=1}^{m} (f(-\frac{1}{2\pi i} \lambda_j))$

EXAMPLE:

What is the Chern f-genus associated to

$$f(z) = (1+z)^{-1}$$

$$\Pi_f(\underset{(-2\pi i)}{X}) = \prod_{j=1}^{m}(1+\lambda_j) = \prod_{j=1}^{m}(1-\lambda_j+\lambda_j^2-\lambda_j^3+\dots)$$

$$= 1-(\underbrace{\lambda_1+\lambda_2+\dots}_{c_1(X)}) + \underbrace{(\lambda_1^2+\lambda_2^2+\dots+\lambda_1\lambda_2+\dots)}_{c_1^2-c_2}+\dots = 1-c_1(X)+c_1^2-c_2+\dots$$

If $E_1 \oplus E_2$ is trivial

$$c(E_1)c(E_2) = c(E_1\oplus E_2) = 1$$

$$= \det\left(1-\frac{1}{2\pi i}F^{E_1}\right)\det\left(1-\frac{1}{2\pi i}F^{E_2}\right)$$

$$\Rightarrow c(E_2) = \det\left(\left(1-\frac{1}{2\pi i}F^{E_1}\right)^{-1}\right)$$

chern genus of $f=(1+z)^{-1}$

$\Rightarrow$ We know that

$$c_1(E_2) = c_1(E_1)$$

$$c_2(E_2) = c_1^2(E_1) - c_2(E_1)$$

can also get this directly.

Now consider a real vector bundle $E \longrightarrow M$, so an $O(n)$ bundle.

Let's define characteristic classes for E in terms of characteristic classes of $E\otimes\mathbb{C} \longrightarrow M$

Note:

$$c_k(E\otimes\mathbb{C}) = \mathrm{tr}\left((-2\pi i)^k \Lambda^k F_\nabla\right)$$

F_∇ is in $\mathfrak{O}(n)$, so $F_\nabla^T = -F_\nabla$

Then, $tr((-2\pi i)^k \Lambda^k F_\nabla^T) = (-1)^k tr((-2\pi i)^k \Lambda^k F_\nabla)$

So if k is odd, $c_k(E \otimes \mathbb{C}) = 0$

We define pontryagin classes of E to be the even Chern classes of $E \otimes \mathbb{C}$

$$p_k(E) = c_{2k}(E \otimes \mathbb{C}) \in H^{4k}(M)$$

In a similar fashion for chern genus, we can define pontryagin genus.

Let g be holomorphic near 0 and $g(0) = 1$

Let f be the branch of

$$z \longmapsto (g(z^2))^{1/2}$$

which has $f(0) = 1$

Note: f is even, so the Chern f genus involves only even chern classes.

DEFINITION: With f as above, the pontryagin g-genus of E is the Chern f-genus of $E \otimes \mathbb{C}$.

LEMMA Let g be as above, then for a real bundle the pontryagin g-genus is equal to

$$\prod_j g(y_j)$$

2×2 blocks in antisymmetric matrix.

where y_j = "eigenvalues", for which pontryagin classes are the elementary symmetric polynomials.

"Taking an orthogonal matrix & diagonalizing as a complex matrix"

$X \in O(n)$ is similar $\overset{\text{over } \mathbb{R}.}{\text{to}}$ a block diagonal

$$\begin{bmatrix} \Lambda_1 & & & 0 \\ & \Lambda_2 & & \\ & & \ddots & \\ 0 & & & \Lambda_{l.o.} \end{bmatrix} \quad \text{where} \quad \Lambda_j = \begin{bmatrix} 0 & -\lambda_j \\ -\lambda_j & 0 \end{bmatrix}$$

y_j's are just λ_j's

can have zeros

Proof of lemma comes down to

$$p_1\left(\begin{bmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{bmatrix}\right) = -\frac{\lambda_j^2}{4\pi^2}$$

whereas, over $\mathbb{C}$

$$\begin{pmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{pmatrix} \text{ is similar to } \begin{pmatrix} -i\lambda_j & 0 \\ 0 & \lambda_j \end{pmatrix}$$

$$c_1 = 0, \qquad c_2 = \frac{1}{(2\pi i)^2}(-i\lambda_j)(i\lambda_j) = -\frac{\lambda_j^2}{4\pi^2}$$

UNMOTIVATED EXAMPLES (HOPEFULLY IMPORTANT)

The $\hat{A}$ genus of a vector bundle is associated to

$$g(z) = \frac{\sqrt{z}/2}{\sinh\left(\frac{\sqrt{z}}{2}\right)}$$

$$= 1 - \frac{z}{24} + \frac{7z}{5760} + \cdots$$

Then,

$$\hat{A}_1 = -\frac{p_1(E)}{24}$$

Wikipedia tells us that $\hat{A}_2 = \frac{1}{5760}(-4p_2 + 7p_1^2)$

The Hirzebruch's genus is associated to

$$z \longmapsto \frac{\sqrt{z}}{\tanh\sqrt{z}}$$