Characteristic classes.- Physics seminar - February 2017

Goal: discuss index thms.

Consider a $\overset{\text{(complex)}}{}$ r vector bundle $E \to M$, a connection is a map
$$\nabla : \Gamma(TM) \times \cancel{\Gamma(E)}E) \longrightarrow \Gamma(E)$$
that satisfyies: $\nabla(X, s) \longmapsto \nabla_X s$

Is C^∞-linear in X, $\mathbb{C}$-linear in s and satisfies the Leibnitz rule:
$$\nabla_X (fs) = (d_X f) s + f \nabla_X s$$

In a local trivialization, s is just a $\mathbb{C}^r$ valued functions
(r = rank of the bundle)
and then
$$\nabla s = ds + As \quad \text{where } A \text{ is a matrix valued 1-form}$$
$$\nabla_X s = ds(X) + A_X s.$$
A is called the connection form, gauge potential.

Given connections ∇^1, ∇^2 on E^1, E^2, we can get the following induced connections on:

$E^1 \otimes E^2$ has $(\nabla^1 \otimes \nabla^2)(s^1 \otimes s^2) = \nabla^1 s^1 \otimes s^2 + s^1 \otimes \nabla^2 s^2$

∇^{Hom} on $Hom(E^1, E^2)$

If $\Phi : E^1 \to E^2$ i.e Φ_x is a linear map $E_x^1 \to E_x^2$

$$(\nabla^{Hom}\,\Phi)\,s' = \nabla^2(\Phi s') - \Phi(\nabla' s')$$

this is again just enforcing a product rule

We also use connections to define exterior covariant differentiation d_∇ $\qquad \Omega^p(E) = \Gamma(\Lambda^p T^* M \otimes E)$

$$d_\nabla : \Omega^p(E) \longrightarrow \Omega^{p+1}(E)$$

where $\Omega^p(E) = \Omega^p(M) \otimes \Gamma(E)$, and locally we just get usual $\Omega^p(M)$ & d_∇ is usual exterior derivative.

For $\omega \in \Omega^k(M)$ and $B \in \Omega^\ell(E)$

$$d_\nabla(\omega \wedge B) = d\omega \wedge B + (-1)^k \omega \wedge d_\nabla B$$

for $s \in \Gamma(E) = \Omega^0(E)$ $\quad d_\nabla s = \nabla s$

$B = \eta \otimes f \quad \eta \in \Omega^s(M)$ & $f \in \Gamma(E)$

$$\Rightarrow \quad d_\nabla B = d\eta \otimes f + (-1)^k \eta \wedge df$$

Unlike the exterior derivative, in general, it is <u>not</u> true that $d_\nabla^2 = 0$.

In fact $\exists$ $F_\nabla \in \Omega^2(End(E))$ s.t.

$$d_\nabla^2 s = F_\nabla s$$

$F_\nabla = F_{ij}\, dx^i \wedge dx^j$

$F_{ij} \in End(E_x)$

F_∇ is called the curvature.

Facts $F_\nabla(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$

for $X, Y \in \Gamma(TM)$.

Locally, $F_\nabla = F_{ij}\, dx^i \wedge dx^j$

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$

$$A = A_i\, dx^i$$

Under change of trivialization (gauge transformation) g

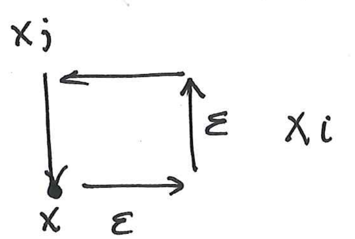
$$A \longrightarrow g\,dg^{-1} + g A g^{-1}$$

$$s \longrightarrow g s$$

$$F \longrightarrow g F g^{-1}$$

Use curvature to define topological inv. for bundle
Not obvious that this is the case.
Before defining characteristic classes, consider the following

Fact: In local coords, consider parallel translation around square in $x_i x_j$-plane w/ side length ε.



This gives a map

$$\Phi_{ij} : E_x \to E_x$$

$$\Phi_{ij} = 1 + \varepsilon^2 F_{ij} + O(\varepsilon^4)$$

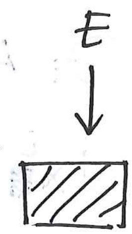
curvature tells us how "bad" parallel trans. is from getting Id.

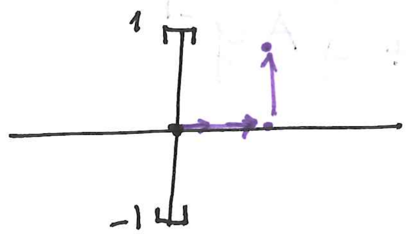
PROPOSITION Let $M = [-1,1]^n$. Let E be a vector bundle & with connection ∇. If $F_\nabla \equiv 0$, then E is trivial with trivialization in which $A = 0$

(Note M contractible $\Rightarrow$ E trivializable (top))

Proof

$(n=2)$ Fix a basis at E_0, extend to a basis on x^1 axis by parallel translation Then parallel translate $\boxtimes$ along x^2 to get basis at every E_x





we did not use $F_\nabla \equiv 0$ yet!

$$F_{12} = 0 = \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]$$

$A_1 = 0$ along x^1 axis

$A_2 = 0$ everywhere

$\Rightarrow F_{12} = -\partial_2 A_1 = 0 \Rightarrow A_1 = 0$ everywhere

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This prop was just over cubes, but something is happening top. zero curvature enforces triviality for some gauge

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Corollary: If M is simply connected & $F = 0$, then E is trivial (& get $A = 0$

For the rest of the talk: Specify structure group to be either $U(n)$ or $SU(n)$ if vector bundle is complex or $O(n)$ or $SO(n)$ if vector bundle is real

"Restricting to changes of trivializations in these groups"

If our structure group is some lie group G, then our connection for A is locally $\mathfrak{g}$ valued.

Recall $A \rightsquigarrow g\,dg^{-1} + gAg^{-1}$

For a bundle $E_{\mathbb{C}}$ with connection ∇, define

$$c_1(E) = tr\left(\frac{i}{2\pi} F_\nabla\right) \in \Omega^2(M)$$

$$= tr\left(\frac{i}{2\pi} F_{jk}\right) dx^j \wedge dx^k$$

locally a $u(n)$, $su(n)$ valued matrix.

If $F_{ij} \in u(n)$ then diagonal is in $i\mathbb{R}$

So we get $tr\left(\frac{i}{2\pi}\right) F_\nabla$ to be a <u>real</u> valued 2-form.

Here $u(n)$ is anti Hermitian matrices.

Recall, the Bianchi identity

$$d_\nabla F_\nabla = 0$$

components of this are $[A_i, F_{jk}]$

traces of commutators are zero

$$d_\nabla F_\nabla \underset{(loc)}{=} dF_\nabla + \overbrace{A \wedge F_\nabla - F_\nabla \wedge A}$$

where we have matrix mult in matrix part & wedge in form part

add it in for free

$$d\, tr\left(\frac{i}{2\pi} F_\nabla\right) = tr\left(\frac{i}{2\pi} dF_\nabla + A \wedge F_\nabla - F_\nabla \wedge A\right)$$

$$= tr\left(\frac{i}{2\pi} d_\nabla F_\nabla\right) = tr(0) = 0$$

$\Rightarrow \quad c_1(E) \in H^2_{dR}(M)$

<u>CLAIM</u> As a class in $H^2_{dR}(M)$, $c_1(E)$ does not depend on ∇.

<u>Proof</u>: Fix a connection ∇ on E, then another connection $\tilde{\nabla}$ can be written as

$$\tilde{\nabla} = \nabla + B \quad \text{for some } B \in \Omega^1(\mathrm{End}(E)).$$

$$\Rightarrow F_{\tilde{\nabla}} = F_\nabla + d_\nabla B + B \wedge B \qquad \text{(exercise).}$$

$$= F_\nabla + dB + B\wedge A + A \wedge B + B \wedge B \quad \text{local}$$

$$\Rightarrow \operatorname{tr}\left(\frac{i}{2\pi} F_{\tilde{\nabla}}\right) = \operatorname{tr}\left(\frac{i}{2\pi} F_\nabla\right) + \operatorname{tr}\left(\frac{i}{2\pi}\left(dB + B\wedge A + A \wedge B + B \wedge B\right)\right)$$

$$= \operatorname{tr}\left(\frac{i}{2\pi} F_\nabla\right) + \underbrace{d \operatorname{tr}\left(\frac{i}{2\pi} B\right)}$$

global form

IMPORTANT!

So they're equal in the cohomology!

Because $F \rightsquigarrow gFg^{-1}$ $\quad \operatorname{tr} F = \operatorname{tr}(gFg^{-1})$

Can any $[C] \in H^2_{dR}(M)$ be $c_1(E)$ for some E?

We call $c_1(E)$ the first chern class of E!

Characteristic classes:

given a vector bundle associate to it an element $[C] \in H^2(M)$.

Given $X \in gl_m(\mathbb{C})$, X acts on $\mathbb{C}^m$ & induces a map $\Lambda^k X$ acting on $\Lambda^k \mathbb{C}^m$

$$\Lambda^k X(v_1 \wedge \dots \wedge v_k) = Xv_1 \wedge Xv_2 \wedge \dots \wedge Xv_k$$

We then define

$$c_k : gl_m(\mathbb{C}) \longrightarrow \mathbb{C}$$

$$c_k(X) = (-2\pi i)^{-k} \operatorname{tr}(\Lambda^k X)$$ with trace on $\Lambda^k \mathbb{C}^m$

For X diagonal,

$c_k(X)$ is just the kth elementary symmetric polynomial on the diagonal entries

Useful fact:

$$\det\left(1 + \frac{it}{2\pi} X\right) = \sum_k t^k c_k(X)$$

Note that the left hand side is $GL_m\mathbb{C}$ invariant

$$\det\left(1 + \frac{it}{2\pi} gXg^{-1}\right) = \det\left(1 + \frac{it}{2\pi} X\right)$$

So right hand side is also $GL_m \mathbb{C}$ inv.

We say a polynomial $P : gl_m \mathbb{C} \to \mathbb{C}$ is invariant if $P(gXg^{-1}) = P(X)$ $\forall g \in Gl_m \mathbb{C}$

P could also be a formal power series

FACT: $\{c_k(X)\}$ generate the set of invariant polynomials on $gl_m \mathbb{C}$.

For any X with distinct eigenvalues, then X is diagonalizable, gXg^{-1} diagonal for some g and

$$P(X) = P(gXg^{-1})$$

must be a symmetric polynomial in the eigenvalues. By continuity argument we get P symmetric in eigenvalues, b/c set of such X in space of matrices is dense.

locally F_∇ is a matrix valued 2-form, so define

$$c_k(E) := c_k(F_\nabla)$$

Invariance of $c_k \Rightarrow c_k(E)$ is well defined & in fact won't depend on ∇

We call $c_k(E)$ the kth chern class of E.

PROPOSITION: $d\,c_k(E) = 0$, and as an element of $H^{2k}_{dR}(M)$, $c_k(E)$ does not depend on ∇

$c_k(E)$ homogeneous pol of deg k, & b/c F_∇ two forms we get something in $H^{2k}(M)$.

"Proof"

We can think of c_k as a linear map

$$\underbrace{\mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{k} \longrightarrow \mathbb{C}$$

fact that it's invariant induces a bundle morphism

$$\varphi : End(E) \otimes \dots \otimes End(E) \longrightarrow \mathbb{C}$$

invariance $\Rightarrow$ constant in every frame
$\Rightarrow$ covariant constant

$$\begin{aligned} d\, C_k(E) &= d\, \varphi(F_\nabla, \dots, F_\nabla) \\ &= \varphi(d_\nabla(F_\nabla \otimes \dots \otimes F_\nabla)) \\ &= \varphi(d_\nabla F_\nabla \otimes F_\nabla \otimes \dots \otimes F_\nabla) + \dots \\ &+ \varphi(F_\nabla \otimes \dots \otimes F_\nabla \otimes d_\nabla F_\nabla) \\ &= 0 \quad \text{by Bianchi identity} \end{aligned}$$

Well definedness comes from fact that $(1-t)\nabla^0 + t\nabla^1$ defines homotopy from ∇^0 to ∇^1 $\square$

If E trivial $\quad C_k(E) = 0 \quad \forall\, k \quad (k \neq 0)$

If $C_k(E) = 0 \quad \forall\, k$, E may still not be trivial. especially if M not simply connected

<u>RMK</u> If structure group is $SU(n)$ then $F_{ij} \in SU(n)$ is traceless, so $C_1(E) = 0$

How do you work with chern classes? think of them as elem sym polynomials in the eigenvalues & that allows us to compute easily

e.g X diagonal with entries λ_j

$$tr(X^2) = (\lambda_1^2 + \lambda_2^2 + \dots + \lambda_m^2)$$

normalize!

$$\operatorname{tr}\left(\left(\frac{i}{2\pi} X\right)^2\right) = \frac{-1}{4\pi^2}\left(\lambda_1^2 + \cdots + \lambda_m^2\right)$$

$$= c_1^2(\lambda) - 2c_2(\lambda)$$

$$\Rightarrow \operatorname{tr}\left(\left(\frac{i}{2\pi} F_\nabla\right)^2\right) = c_1^2(E) - 2c_2(E).$$

In particular if $G = SU(n)$

$$\Rightarrow c_2(E) = \operatorname{tr}\left(\frac{i}{8\pi^2} F_\nabla^2\right)$$

Important fact to show self dual connections minimize the l_2 norm of the curvature.