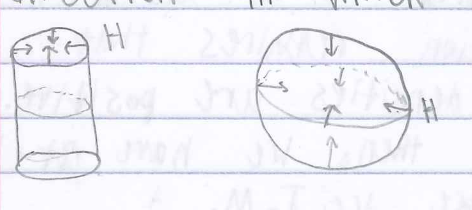
10/30/19

Day 4: Singularity Theorems

We've spent a lot of time discussing what singularities are, but we've yet to convince ourselves that this is a worthwhile pursuit for practically understanding the universe — it may well be that physically reasonable GR models are generically not singular. The only physically reasonable singular spacetimes we've encountered, after all, are the Schwarzschild and FLRW models, both of which invoke an unreasonably high degree of symmetry. While we may expect these symmetries to be approximately present, if the singular nature of these models depends crucially on the symmetry, then the singularities would disappear even for arbitrarily "close" spacetimes. The singularity theorems ensure us this is not the case — the singular behavior of near-Schwarzschild and near-FLRW spacetimes is generic.

To qualitatively understand these theorems, we look back at some geometric concepts. Given a submanifold $N \subset M$ of (M, g), g naturally induces a metric $\tilde{g}$ on N with which we can describe the intrinsic geometry of N — properties of N as a manifold with metric in its own right independent of its embedding in (M, g). However, we can also ask about the geometry of how N sits in M. This is described by the so-called second fundamental form, from which we can define the mean curvature vector field H, which is everywhere $\perp$ to N and roughly indicates the average direction in which N is curving in M.



When H is nonzero, geodesics emanating from and $\perp$ to N with the property $\langle \gamma'(0), H \rangle > 0$ tend to converge and focus at a point. The larger the value of $\langle \gamma'(0), H \rangle$, the faster the geodesics converge.

This observation is crucial to the singularity theorems and leads us to the following: if M is Lorentzian and N is spacelike and codimension 1, H is timelike (being $\perp$ to the $n-1$ dimensional spacelike subspace T_pN of T_pM at each $p \in N$) and we define the convergence $\kappa := \langle U, H \rangle$, where U is the future-pointing unit normal to N. When $\kappa > 0$, geodesics normal to N tend to converge.

These observations, however, depend on one further hypothesis. For geodesics to converge, we need not only that N be shaped to focus them, but that M is not curved so as to repel the geodesics from each other. Recall the geodesic deviation equation (or Jacobi equation) for the infinitesimal displacement J between nearby geodesics,

$$\nabla_{\gamma'} \nabla_{\gamma'} J + R(\gamma', J)\gamma' = 0.$$

So the component of the acceleration of J $\nabla_{\gamma'}\nabla_{\gamma'} J$ in the J direction is $-\langle R(\gamma', J)\gamma', J \rangle = -R(\gamma', J, \gamma', J)$.

If $R(\gamma', \cdot, \gamma', \cdot)$ is positive on average, then we expect that geodesics near γ will not be pulled apart. This averaging is exactly done by contracting the 2nd and 4th indices, so our desired condition is $Ric(\gamma', \gamma') \geq 0$. Is this reasonable?

Recall Einstein's equation, $Ric - \frac{1}{2} R g = 8\pi T$, and further that $T(v, v)$ is physically the amount of v-momentum flowing in the v direction. In particular, if v is timelike $T(v, v)$ is energy density as measured by an observer with velocity v. The <u>weak energy condition</u> requires that $T(v, v) \geq 0$, i.e. that measured energy densities are positive. Under this (very reasonable!) condition, then, we have at least $Ric(v, v) - \frac{1}{2} R g(v, v) \geq 0$ for any timelike $v \in T_pM$.

Going a bit further, we might require the <u>strong energy condition</u>, that $T(v,v) - \frac{1}{2}\operatorname{tr}(T)\, g(v,v) \geq 0$ (notice $g(v,v) < 0$ for v timelike). This is, by the Einstein equation, equivalent to $Ric(v,v) \geq 0$. The strong energy condition, then, is a physically motivated condition that ensures that, on average, geodesics don't tend to diverge under the influence of gravity alone.

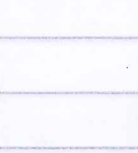
Thus, under this condition, when N has positive convergence geodesics normal to N should focus. If $q \in M \setminus N$, one can define $\tau(N,q)$ as the maximal proper time of a timelike curve connecting N to q. If N is, in addition, a Cauchy hypersurface, one can show that $\tau(N,q)$ is realized as the proper time of a timelike geodesic γ initially normal to N. The crucial fact used in Hawking's singularity theorem is that, if a family of geodesics focuses at a point along γ before q, then γ cannot give the maximal proper time! If we know from a convergence condition that γ must have a focal point within a time b, then, this shows every timelike curve from N to q has proper time less than b!

More precisely, Hawking's Singularity Theorem (1967) states: If $Ric(v,v) \geq 0$ for every timelike tangent vector to M and N is a spacelike Cauchy hypersurface with future convergence $K \geq b > 0$, then every future-pointing timelike curve starting in N has proper time at most $1/b$.

Hawking also proved a weaker conclusion, future timelike geodesic incompleteness, under the modified hypotheses that N is only a compact spacelike hypersurface (not necessarily Cauchy!) with future convergence $K > 0$ (no a priori lower bound).

The time-reversed statements of these theorems are seen as proofs that the universe almost certainly began with a big-bang-type singularity (as far as GR is concerned), because, in the FLRW model, slices of constant time satisfy the role of N in the theorem, where the convergence is directly measurable as the Hubble parameter $\frac{\dot{a}}{a}$! These key elements of the hypotheses are not dependent on the symmetry of the FLRW model, so it is expected that the "actual" spacetime still has such a Cauchy hypersurface, and our measurements of the expansion of the universe strongly suggest the relevant convergence is bounded away from 0.

Only a couple of years before, in 1965, Penrose proved his singularity theorem more relevant to Schwarzschild-type singularities. His theorem also uses the notion of convergence, but for a codimension 2 submanifold. Without a unique unit normal, now we wish to require that $k(v) := \langle H, v \rangle > 0$ for every future-pointing null vector, which is equivalent to requiring that H be past-directed timelike. In this case, we call N *future-converging*, as now future-directed null curves through N tend to be focused. A key sub-result of Penrose's theorem was that this condition, together with with the weak energy condition, implies that the future of N is confined to a "small" region in the sense that $E^+(N) := J^+(A) \setminus I^+(A)$ is compact if M is future null complete.



Penrose's full result is as follows:

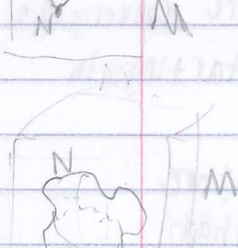
If (1) the weak energy condition holds

(2) M has a Cauchy hypersurface (is globally hyperbolic)

(3) N is a compact achronal spacelike codimension 2 future-converging submanifold

(4) M is future null complete

Then $E^+(N)$ is a Cauchy hypersurface in M.



We obtain a singularity result from this by observing, then, that if (1), (2), and (3)