Dirac

08/28/19

Day 1: Manifolds, tangent space, tensors, metrics, connections, ~~curvature~~

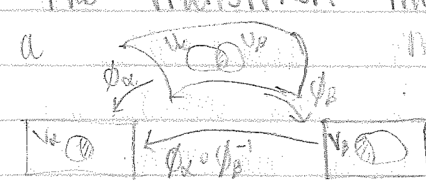
For our purposes, a topology on a set can be thought of as "the minimal structure necessary to take limits" (nets/sequences), provided by a collection of subsets dubbed "open".

An n-dim. manifold M is a topological space (a set + topology) (Hausdorff, 2nd countable) in which every point has a nbhd that "looks like $\mathbb{R}^n$", i.e. $\forall\, p \in M\ \exists$ open sets $U \subseteq M$, $V \subseteq \mathbb{R}^n$ and a homeomorphism $\phi: U \to V$.

Reasonable model for reality: we can always use a ruler and a stopwatch to put coordinates on our surroundings in the set of "events". Importantly, this is a local observation: doesn't say much about global topology.

Reality tells us more: that physical systems seem to satisfy differential equations describing evolution consistent between coordinate systems of different observers (e.g., Lorentz invariance of Maxwell's equations), we expect additional structure allowing consistent use of calculus.

A manifold is smooth if it can be covered by coordinate systems $(U_\alpha, V_\alpha, \phi_\alpha)$ (recall $U_\alpha \subseteq M$, $\phi_\alpha: U_\alpha \to V_\alpha$) s.t. when two overlap, say $U_\alpha \cap V_\alpha \neq \emptyset$, the transition map $\phi_\alpha \circ \phi_\beta^{-1}: V_\beta \to V_\alpha$ is smooth (as a map $\mathbb{R}^n \to \mathbb{R}^n$).



Easy to reduce to differentiable, C^k, etc.

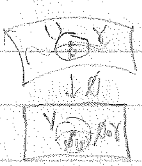
As we would want, this structure guarantees we may do calculus in different coordinate systems consistently.

The equivalence principle asserts the equality of inertial and gravitational mass, implying that the motion of a (suff. small) object under only gravity is independent of the nature of the object, i.e. its mass, charge, shape, etc. One interpretation is that gravity then apparently identifies a family of curves in the spacetime manifold: the curves followed by "test particles" with any given initial pos. + vel.

Main idea of GR: Suppose we interpret this family of curves as a geometric property of the spacetime manifold M, so that these are the "default" curves on M. What geometric structures would we need to identify such curves? We'd like to say there's a sense in which they're not accelerating w.r.t. M. This takes some work.

First problem: need a notion of the velocity of a curve which is a smooth map $\gamma: I \to M$, where $I \subseteq \mathbb{R}$ is an interval. $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$

This is easy in $\mathbb{R}^n$, where γ has coordinate components which can be differentiated to yield components of a new vector, the tangent vector γ'. In M, we may do something similar by projecting to coordinates around any point and computing $(\phi \circ \gamma)'$. This is extremely messy: every coordinate system at p assigns a tangent vector in a different vector space (cannot add a result from one coordinate system to that from another).



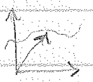
An approach that yields a single vector space manifestly independent of coordinate systems is associating to each curve γ and $t_0 \in I$ a "directional derivative" on M at $\gamma(t_0)$, i.e. the map $C^\infty(M) = \{f: M \to \mathbb{R} \mid f \text{ smooth}\} \to \mathbb{R}$ given by $f \mapsto (f \circ \gamma)'(t_0)$ (the derivative of f in the direction of γ). The <u>tangent space</u> T_pM to M at $p \in M$ is then the vector space of "directional derivatives", or derivations at p, $\delta: C^\infty(M) \to \mathbb{R}$, $\delta(\alpha f + \beta g) = \alpha\delta(f) + \beta\delta(g)$, $\delta(f \cdot g) = f(p)\delta(g) + \delta(f) \cdot g(p)$. One can show that every derivation arises as a directional derivative along some curve and vice-versa. The <u>tangent vector</u> to γ at t_0 is denoted by $\gamma'(t_0) \in T_{\gamma(t_0)}M$ and is given as above by $f \mapsto (f \circ \gamma)'(t_0)$.

Given a coordinate system around $p \in M$ with coordinates $x_1, \ldots, x_n$ (i.e. $x_1, \ldots, x_n$ are the coordinate functions of $\mathbb{R}^n$ on $V = \phi(U)$), then each coordinate clearly has an associated directional derivative: the partial derivative $f \mapsto \frac{\partial}{\partial x_i}(f \circ \phi^{-1})$, denoted by $\frac{\partial}{\partial x_i}$ or $\partial_i \in T_pM$; ∂_i is the tangent vector at p to the curve with x_i changing and all other x_j constant, $\gamma_i(t) = \phi^{-1}((x_1(p), \ldots, x_{i-1}(p), t, x_{i+1}(p), \ldots, x_n(p))$. These form a basis of T_pM, and can be translated between coordinate systems as $\frac{\partial}{\partial x_i} = \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}$

A <u>vector field</u> X on M, then, is an association to each $p \in M$ a vector $X_p \in T_pM$. Typically assumed smooth (i.e. given a coordinate system, the expansion $X = X^i \partial_i$ has smooth component functions X^i). By the derivative interpretation of tangent vectors, vector fields naturally act on functions $f \in C^\infty(M)$ to give new functions $X(f) \in C^\infty(M)$ via $X(f)(p) = X_p(f)$

w.r.t. M

Second problem: To claim a curve is "not accelerating", we need a way to say its velocity vector is unchanging as one moves along the curve. Again, this is simple in $\mathbb{R}^n$, where one may differentiate the components of γ twice to obtain its acceleration



$\gamma'' = (\gamma_1'', \dots, \gamma_n'')$ $\gamma''(t)$. (or $\gamma''(t)_i$). No good way to do this in general: to differentiate a vector field, as when one attempts to evaluate something like $\lim_{t \to t_0} \frac{X_{\gamma(t)} - X_{\gamma(t_0)}}{t - t_0}$, it is unclear what the numerator should mean — at what point would it be a directional derivative?

However, we can enumerate some fundamental properties that a "derivative of a vector field" in some direction should have:

An <u>(affine) connection</u> on M is a bilinear map $\nabla : \mathrm{Vec}(M) \times \mathrm{Vec}(M) \to \mathrm{Vec}(M)$, written $(X, Y) \mapsto \nabla_X Y$, satisfying

(1) $\nabla_{f \cdot X} Y = f \cdot \nabla_X Y$ for $f \in C^\infty(M)$

(2) $\nabla_X (fY) = X(f) \cdot Y + f \cdot \nabla_X Y$

$\nabla_X Y$ is interpreted as a "derivative of Y in the direction of X".

$\boxed{\text{Ex}}$ In $\mathbb{R}^n$, $(\nabla_X Y)_i = X \cdot \vec{\nabla} Y_i$

If a connection ∇ on M is provided, then, we may reasonably say that a curve γ is not accelerating if $\nabla_{\gamma'} \gamma' = 0$, i.e. "the derivative of γ' in the direction of γ' is zero". γ is then called a <u>geodesic</u> of ∇.

Given a manifold, though, there are infinitely many different connections giving infinitely many different families of geodesics. To motivate the choice of the "correct" one, we need another physical observation.

First, a quick aside:

$\in T^k_q M$

A <u>(k,q)-tensor</u> on M at $p \in M$ is a (multi-) linear map $(T_pM)^{\otimes q} \to (T_pM)^{\otimes k}$. Such a tensor T then takes in q tangent vectors $v_1, \dots, v_q$ and returns an element $T(v_1, \dots, v_q)$ of the kth self tensor product of T_pM. We will usually deal in the cases $\cancel{k} = 0, 1$, so T returns either a number or a tangent vector. Naturally we have <u>(k,q)-tensor fields</u> (a (k,q)-tensor S_p at each point $p \in M$). $S(\partial_{i_1}, \dots, \partial_{i_k}) = S^{j_1 \cdots j_q}_{i_1 \cdots i_k} \partial_{j_1} \otimes \partial_{j_2} \cdots \otimes \partial_{j_q}$ in coordinates.
Importantly, since tensor fields act point-wise — $S(X,Y)(p) = S_p(X_p, Y_p)$ — tensor fields are (multi-) linear w.r.t. functions: $S(f \cdot X, Y) = f \cdot S(X,Y)$, etc.
In particular, a connection is <u>not</u> a $(1,2)$-tensor field.

Now, physically we observe that on small enough scales, special relativity describes classical physics well, i.e., any point should have a nbhd and a set of "inertial observers" at that point for which their stopwatch + ruler coordinate system has the property that the "preferred" curves of gravity are (nearly) straight lines, and further the translation between these coordinate systems are the Lorentz transformations,

travelling along inertial curves

$$t' = \gamma \cdot (t - vx) \qquad \left(\gamma = \frac{1}{\sqrt{1-v^2}}\right)$$
$$x' = \gamma \cdot (x - vt)$$
$$y' = y \qquad\qquad x = \alpha t \Rightarrow x' = \gamma(\alpha - v)t = \frac{(\alpha - v)}{(1-\alpha v)} t' \qquad t' = \gamma(1-\alpha v)t$$
$$z' = z$$

The salient point: Lorentz transformations preserve the spacetime interval $\Delta s^2 := (\Delta \vec{x})^2 - (\Delta t)^2 = (t\ x\ y\ z) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$
Since Lorentz transformations are linear, they transform coordinate descriptions of tangent vectors to curves in the same way as the coordinates, implying that the $(0,2)$ tensor g_p at a point p described in some inertial coordinates by $\begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}$, i.e. $g_p(V, W) = g_p(V^i \partial_i, W^j \partial_j) = g_p(\partial_i, \partial_j) V^i W^j = (g_p)_{ij} V^i W^j = -V^0 W^0 + \sum_{i=1}^{3} V^i W^i$, has this same presentation according to all inertial observers at p. That is, apparently M is endowed with a symmetric, nondegenerate $(0,2)$-tensor of Lorentzian signature that is physical in the sense that every inertial coordinate system at a point diagonalizes it there. Assuming this tensor is smooth, this is a

Lorentzian metric.

Our current scenario, then, is that physical observations motivate modeling classical reality via a smooth 4-manifold with a Lorentzian metric somehow associated to gravity, and we'd like a way to identify a connection ∇ with respect to which geodesics are the paths followed by test particles under the influence of gravity.

Most straightforward way to proceed:
Given a metric g on a manifold, there is a unique connection $\bar{\nabla}$ called the Levi-Civita connection, satisfying the following properties:
(1) Metric compatibility: $Z(\langle X, Y\rangle) = \langle \bar{\nabla}_Z X, Y\rangle + \langle X, \bar{\nabla}_Z Y\rangle$
(2) Torsion free: $\bar{\nabla}_X Y - \bar{\nabla}_Y X = [X, Y]$
Of these, (1) most immediately has a necessary physical consequence:
If γ is a geodesic of $\bar{\nabla}$, then

$$\frac{d}{dt}(\langle \gamma'(t), \gamma'(t)\rangle) = \gamma'(t)(\langle \gamma', \gamma'\rangle) = 2\langle \nabla_{\gamma'}\gamma', \gamma'\rangle = 0$$

So $\langle \gamma'(t), \gamma'(t)\rangle$ doesn't change along geodesics. In particular, this implies that test particles initially not going faster than light can never accelerate to light speed under the influence of gravity according to any inertial observer at any point.
(2) at this point seems more an ad hoc requirement to achieve uniqueness, though it does have the nice consequence that geodesics extremize proper time, i.e. the functional
$\tau[\gamma] = \int \sqrt{-\langle \gamma'(t), \gamma'(t)\rangle}\, dt$

Given this approach, gravity is completely described by the metric g. To complete GR as a theory of gravity, it remains to establish the relationship between g and the matter distribution of the universe. This is the content of Einstein's equation, to be discussed by Ben as the equation of motion extremizing the Einstein-Hilbert action in parallel/conjunction with a discussion of extremizing a similar action in the presence of torsion. To set the stage for the latter discussion, we present one last topic.

A general connection can be characterized in a coordinate system by its action on the coordinate basis, i.e.

$\nabla_{\partial_i}\partial_j = \Gamma_{ij}{}^k \partial_k$ ($\Gamma_{ij}{}^k$ called the Christoffel symbol of ∇), since

$\nabla_X Y = \nabla_{X^i\partial_i} Y^j \partial_j = X^i \nabla_{\partial_i} Y^j \partial_j = X^i \partial_i(Y^j)\partial_j + X^i Y^j \Gamma_{ij}{}^k \partial_k$.

In coordinates, the geodesic equation reads

$0 = \nabla_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma^i\partial_i}\dot\gamma^j\partial_j = \ddot\gamma^j\partial_j + \dot\gamma^i\dot\gamma^j\Gamma_{ij}{}^k\partial_k \Leftrightarrow \ddot\gamma^k + \dot\gamma^i\dot\gamma^j\Gamma_{ij}{}^k = 0$ for each k.

Apparently the antisymmetric part of $\Gamma_{ij}{}^k$ has no effect on this equation.

(0,3)

The Torsion tensor, T, associated to ∇ is defined as

$T(X,Y,Z) = \langle \nabla_X Y - \nabla_Y X - [X,Y], Z\rangle$ and has coordinate expression

$T_{ijk} = T(\partial_i, \partial_j, \partial_k) = \langle \nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i, \partial_k\rangle = (\Gamma_{ij}{}^l - \Gamma_{ji}{}^l)\cdot\langle \partial_l, \partial_k\rangle = (\Gamma_{ij}{}^l - \Gamma_{ji}{}^l) g_{lk}$.

Raising the third index, $T_{ij}{}^k = \Gamma_{ij}{}^k - \Gamma_{ji}{}^k$. So, the torsion tensor describes the antisymmetric part of $\Gamma_{ij}{}^k$. Consequently, it is often said that "adding torsion does not change geodesics". In the case that one modifies a symmetric connection by $\nabla \longrightarrow \nabla + T$, this statement is indeed true.

In the presence of a metric, we may also define the (0,3) metric compatibility tensor associated to ∇,

$M(X,Y,Z) = Z(\langle X, Y\rangle) - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle$

Since the L-C connection $\bar\nabla$ satisfies $\bar T = \bar M = 0$, we may write these as

$T(X,Y,Z) = \langle \nabla_X Y - \bar\nabla_X Y - (\nabla_Y X - \bar\nabla_Y X), Z\rangle$

$M(X,Y,Z) = \langle \bar\nabla_Z X - \nabla_Z X, Y\rangle + \langle X, \bar\nabla_Z Y - \nabla_Z Y\rangle$

Combining these, we may describe the (0,3) difference tensor

$$D(X,Y,Z) = \langle \nabla_X Y - \bar\nabla_X Y, Z\rangle$$
$$= \tfrac{1}{2}\{T(X,Y,Z) - T(Y,Z,X) + T(Z,X,Y) + M(X,Y,Z) - M(Y,Z,X) - M(Z,X,Y)\}$$

Remark: this formula implies that ∇ is uniquely determined by T and M.

What happens when we only change torsion? Then $M = 0$ in the above, and

$D(X,Y,Z) = \tfrac{1}{2}\{T(X,Y,Z) - T(Y,Z,X) + T(Z,X,Y)\}$

Importantly, D need not be antisymmetric in its first two indices, even though T is! Thus, the geodesic equation for ∇ is changed relative to that of $\bar\nabla$ by the symmetric part of D.