CONFORMAL PERTUBATION THEORY

Basic Idea: Given a quantum field theory we can perturb it by "couplings" & attempt to define a family of QFTs.

I. A (non)-example: Perturbative QFT

QFT_0 a free theory

= free massless scalar field $\varphi(x)$ in $\mathbb{R}^d$

mass dim $= \frac{d-2}{2}$

has correlation functions

$\langle 1 \rangle = 1$

$\langle \varphi(x) \rangle = 0$

expectation value of a single field

$$\langle \varphi(x)\varphi(y) \rangle = \frac{1}{|x-y|^{d-2}}$$

these determine every other correlation function

Aside conversation:
$\mathbb{Z}_2$ symmetry: Given classical

$L = \partial_\mu \partial^\mu \varphi$ is invariant under $\varphi \to -\varphi$

But we dont want to think this way!

$$\langle \varphi(x)\varphi(y)\varphi(z) \rangle = \langle \varphi(x)\varphi(y) \rangle \langle \varphi(z) \rangle$$

$$+ \langle \varphi(\ldots \quad + \langle \varphi(y)\varphi(z) \rangle . \langle \varphi(x) \rangle = 0$$

$$\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle = \langle \varphi_1 \varphi_2 \rangle \langle \varphi_3 \varphi_4 \rangle + \langle \varphi_1 \varphi_3 \rangle \langle \varphi_2 \varphi_4 \rangle + \langle \varphi_1 \varphi_4 \rangle \langle \varphi_2 \varphi_3 \rangle$$

$\varphi_i = \varphi_i(x)$

with finite amount of pain can compute all correlation functions

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Can also build a $\varphi^2(x)$ operator essentially by a normal ordering procedure

$$:\varphi^2(x): = \lim_{y \to x} \left\{ \varphi(x)\varphi(y) - \frac{1}{|x-y|^{d-2}} \right\}$$

& analogously build $:\varphi^n(x):$

There are also derivatives

$\partial_\mu \varphi, \; \partial_\mu \varphi \partial^\mu \varphi, \; \partial_\mu \partial^\mu \varphi$ → a redundant operator b/c its equal to zero

~~Bcz~~ $\partial_\mu \partial^\mu \varphi(x) = 0$

~~[illegible]~~

NAIVE PERTURBED THEORY:

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_g = \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{-S_{Bare}}$$

where $S_{bare} = \int d^d x \sum_i \lambda_B^i :\varphi^{n_i}(x):$

(★) Compute: $\frac{(-1)^m}{m!} \int d^d y_1 \cdots d^d y_m \lambda_B^{i_1} \cdots \lambda_B^{i_m} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) :\varphi^{n_{i_1}}(y): \cdots :\varphi^{n_{i_m}}(y_m): \rangle$

This is insane! When operators run into each other they have singularities ⇒ ill defined integral!

COMMENTS

[1] Every PQFT is defined this way

[2] No need for a lagrangian - Its suff to have correlators!
but there are aspects of renormalization that become insane w/o a lagrangian though

[3] Naive is nonsense:

Regularize & renormalize.

length cut off a

E.g take (★) and ask that integration domain be $|y_{ij}| \geq a$ $|y_i - x_k| \geq a$

Q: How do I know when I have all local operators in this story?
- Want Operator product expansion closes on operators you have. (e.g. AXIOMs for QFTs) & can take derivatives of correlation functions.
- what about φ^2?

renormalization length scale ℓ

Δ_i = mass dim of deforming operator

$\Delta_i = n_i\left(\frac{d-2}{2}\right)$

$$\lambda_B^i = \ell^{\Delta_i - d}\Lambda_B^i\left(g, \frac{a}{\ell}\right)$$
$$= \ell^{\Delta_i - d} g^i + \mathcal{O}(g^2)$$

$$\Rightarrow S_{Bare} = \int d^d x \sum_i \lambda_B^i : \underbrace{\varphi^{n_i}(x)}_{\Delta_i = n_i\left(\frac{d-2}{2}\right)} :$$

mass dim $d - \Delta_i$

g^i $\begin{cases} \text{is relevant if the } \Delta_i < d \\ \text{marginal if } \Delta_i = d \\ \text{irrelevant if } \Delta_i > d \end{cases}$

FACT: $\exists$ finite # of relevant & marginal caplings & infinite # of irrelevant caplings

Then PT is renormalizable if no irrelevant caplings are turned on

UV divergences absorbed into a finite # of caplings & renormalizations of operators

Q Λ_B - Bare caplings

g stands for $g_1, \dots, g, \dots$

For math people: have a well defined $a \to 0$ limit & doesn't matter if you have diff a's.

Conformal perturbation Theory:
"Start ~~not~~ w/ a free theory, instead w/ some conformal theory and apply same procedure"

"Believed to be true: CPT has a finite radius of convergence (for some extra assumptions)"

|| here means nonzero & finite for math people.

"∃ natural metric on space of caplings CPT around a free theory is like expanding at a pt at inf distance
⇒ get zero radius of conv."

Phrasing CPT in a way that ∃ axiomatic properties that expansion satisfies, then ask is CPT renormalizable?

Formalities $d=2$

Renormalized QFT, QFT depending on caplings g^i, a renormalization scale l, and
finite $\langle \phi_{A_1}(w_1) \cdots \phi_{A_n}(w_n) \rangle$ ϕ_A has mass dim Δ_A

~~suppose~~ which satisfies (*), (**), (***)

CPT

We have CFT_0 primary operators $\hat{\Phi}_A$, can be decomposed

we know their two & 3 point functions

$\langle \phi_A \phi_B \rangle \quad \langle \phi_A \phi_B \phi_C \rangle$

$\Rightarrow$ Define a bare action $S_B = \int d^2 z \{ \lambda^i_B \, \mathcal{O}_i(z, z') \}$

& $\langle \hat{\phi}_{A_1} \cdots \hat{\phi}_{A_n} \rangle = \lim_{a \to 0} \langle \hat{\phi}_{A_1} \cdots \hat{\phi}_{A_n} e^{-S_B} \rangle_{reg}$

renormalized operators

$\hat{\phi}_A = Z_A^B(g, \frac{\ell}{a}) \phi_B$

NOW ASK:

- Is this renormalizable
- Does it respect formal eqns? Yes formally.

- Can you push it to higher orders?
- What's known & what's not known?

* Basic question can you absorb divergences w/ this procedure is not known.

* If you just assume ∃ renormalization schemes special the moduli space of a 2-2 ~~ is kahler this is NOT always true!

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$$(*)\quad \left[-\ell \frac{\partial}{\partial \ell} + \beta^i \frac{\partial}{\partial g^i} \right] \langle \phi_{A_1} \cdots \phi_{A_n} \rangle$$

$$= \Gamma \cdot \langle \phi_{A_1} \cdots \phi_{A_n} \rangle$$

β is a vectorfield

$\beta^i(g)$ = the beta function

matrix of "anomalous $\underrightarrow{\text{dim}}$?" $= \Gamma^{A}_{B}(g)$

$$\Gamma \cdot \langle \phi_{A_1} \cdots \phi_{A_n} \rangle = \Gamma_{A_1}^{B} \langle \phi_B \phi_{A_2} \cdots \phi_{A_n} \rangle$$

$$+ \Gamma_{A_2}^{B} \langle \phi_{A_1} \phi_B(w_2) \cdots \phi_{A_m} \rangle + \cdots$$

(**) Poincaré

(***) The action principal,

$$\frac{\partial}{\partial g^i} \langle \phi_{A_1} \cdots \phi_{A_n} \rangle = B_i\left(g, \frac{\ell}{a}\right) \langle \phi_{A_1} \cdots \phi_{A_n} \rangle - \int_{\text{reg}} d^2 z \langle \mathcal{O}_i(z) \phi_{A_1} \cdots \phi_{A_n} \rangle$$

~~[illegible]~~

defforming operater for g^i

comes from regularization but still has real content

In general $\exists$ singularities so we regulate it

$$[\mathcal{D}_i, \mathcal{D}_j] \neq 0$$

capturing some curvature on space of couplings.