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later : momentum spectrum: Λ — discarded; hidden (in $g(\mu)$); μ

2020. 1.19

RG : study the change of physical system at different scales.

physical system: many coupled d.o.f. within ξ

Group : more precisely, semigroup with identity (in Wilsonian point of view) or monoid.
(group w/o inverse)
like lossy compression.

A little history :

Lorentz : mass renormalization in classical E&M theory
physical mass $m = m_0 + m_{self}$

m_0: mechanical mass $\qquad m_{self}c^2 = \iint \frac{dq\,dq'}{|r-r'|}$

1930s QED and its divergence in higher perturbation expansion.
WWII
1947 ① Shelter Island Conference (first postwar physics conf.)
Lamb : Lamb shift
Kramers : idea of renormalization in non relativistic
Schwinger commented : very good if we lived in a nonrelativistic world
then he proceeded to develop his own relativistic theory.
{ Rabi
{ Schwinger $\frac{\alpha}{2\pi}$
Bethe (reportedly on the train back to Ithaca)
estimated the lamb shift by implementing charge renorm. in the lowest order perturbation theory. (non-relativistic)
1947–1948 Schwinger, Tomonaga : covariant renormalization.
Feynman : path integral and cut-off regularization.
1949 Dyson : show the equivalance.
renormalizability of QED.

Running coupling 1953 & 1954

1.20

1953 Ernst Stückelberg & Petermann (student) Discover the group structure of renormalization. I think it's really a group at their time

(published in a Swiss journal in French, and is little known at that time.) → Helvetica Physica Acta

Other ignored Nobel prize work by Stückelberg

1935 Yukawa model (discouraged by Pauli, didn't publish)

1938 Abelian Higgs model.

(1941 positron as electron travel back in time.)

1943 a long paper outlining complete and correct description for QED, but rejected by Physical Review. ← acknowledged by Feynman

S "They said it was not a paper, it was a programme, an outline, a proposal".

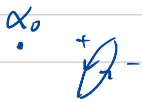
← 1953 by Wilson

...

Gell-Mann always referred to Feynman diagrams as Stückelberg diagram.

1954 Gell-Mann & Low (independently) Introduce the idea of scale transformation in QED and β function (they call ψ) presumabley b/c he dislike Feynman?

Give the physical intepretation of charge renorm.:



virtual pairs screening

pair creation stops at scale m_e, 10^{-13}m.
and $\alpha = \frac{1}{137}$.

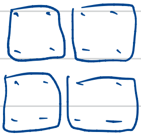
if $m_e = 0$, $\alpha \to 0$ (more precisely $\alpha \sim \frac{1}{\log r}$
and Coulomb potential $\sim \frac{1}{r \log r}$)

(the log correction is common to marginal irrelevant operators.)

near T_c, we can find some scale $a < L \ll \xi \to \infty$

1966 Kadanoff "block spin" transformation.

coarse-graining process.



$f(t,h) = L^{d} f(\tilde{t}, \tilde{h})$ $\quad (h = \beta H)$

assuming $\tilde{h} = L^x h$, $\tilde{t} = L^y t$.

then all critical exponents follows.

conceptually valuable but not computationally

1970 Callan, Symanzik (independently) Callan-Symanzik eq.

reformulation of Gell-Mann & Low.

his advisor

→ principal inspiration: Gell-Mann 1954 and Kadanoff 1966.

1971 Wilson RG, or exact RG equation.

(most clear conceptually, but hard to implement

$$Z = \int_{p^2 \le \Lambda^2} D\phi \, e^{-S_\Lambda[\phi]}$$

integrate over a shell $(\Lambda', \Lambda]$

$$e^{-S_{\Lambda'}[\phi]} := \int_{\Lambda'^2 \le p^2 \le \Lambda^2} D\phi \, e^{-S_\Lambda[\phi]}.$$

Then explore the flow in the space of all couplings

$\Rightarrow$ relevant, irrelevant, marginal.

(actually few)

finite # of relevant $\Rightarrow$ universality.

Put this later

He also pointed out why Landau's theory fails:

Landau: $F = \int d^3x \left((\nabla M(x))^2 + R M^2(x) + U M^4(x) \right)$

R, U should be R_L, U_L, (where the fluctuation $> L$ is integrated out), due to the long range fluctuation.

when $L > \xi$, this is where the fluctuation can be seen as independent, and doesn't change R, U.

$\therefore$ The R, U in Landau theory should be replaced by R_ξ, U_ξ, which introduce new complexity as $T \to T_c$.

1973 Politzer, Gross & Wilczek asymptotically free of YM.

1975 Kadanoff, Houghton Real space RG.

1984 Polchinski improvement: smooth cut-off

easier to calculate.

Another timeline : Critical behavior :

discovered experimentally by Thomas Andrews (1869)

1874 van der Waals : description of liquid-vapor critical point (eq. of state of real gas).

Maxwell gave the equal area construction in the following.

1895 P. Curie pointed out the analogy between magnets and fluids.

1907 Weiss Mean field theory description of the Curie point in magnet.

1934 Bragg & Williams Binary alloys. introduced "Long range order".

1937 Landau generalize to "order parameter" to give a unified description to all second order PT.

Macro emphasized "symmetry".

Micro (L-G) calculate the critical exponents of the MFT

Ginzburg criterion $\langle(\delta\phi)^2\rangle \ll \langle\phi\rangle^2$ universality class.

$\Rightarrow D \geq 2+2\frac{\beta}{\nu}$ Intuition : high dim, reinforced by nbhd.

$\boxed{\text{Wilson's comparison}}$

1941 Kramers & Wannier duality in 2d Ising model.

1944 Onsager 2d Ising model at zero field. (explicitly violate the MFT prediction)

so-called "non-classical universality"

Lots of non-classical exponents were found both theoretically and experimently afterwards.

1965 Widom

scaling hypothesis for free energy near T_c

relationships between critical exponents.

free energy is a homogeneous function of t and h (external field).

(lack of theoretical explanation)

$$H = M^{\delta} h(t M^{-\frac{1}{\beta}})$$

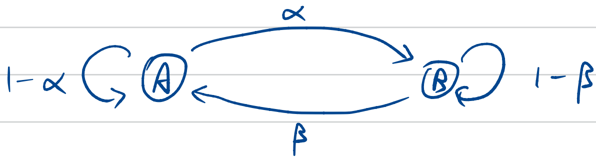
δ, β, $h(x)$ are universal.

Additional notes from Wilson's Nobel lecture:

water-steam transition: the surface tension is like domain wall in spin systems, $\to 0$ at T_c.

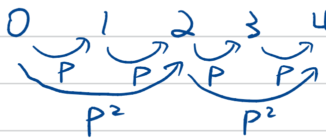
2019.10.31

Example 1 2 state Markov chain.



$$\begin{pmatrix} P_A \\ P_B \end{pmatrix}_{t+1} = \underbrace{\begin{pmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{pmatrix}}_{P} \begin{pmatrix} P_A \\ P_B \end{pmatrix}_t$$

Now this P tells us the physics at time seperation at 1,



P^n tells us the physics at time seperation n.

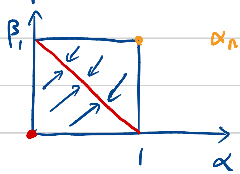
Diagonalize: $P' = \begin{pmatrix} 1 & \\ & 1-\alpha-\beta \end{pmatrix}$

$P'^n = \begin{pmatrix} 1 & \\ & (1-\alpha-\beta)^n \end{pmatrix}$ $\qquad P^\infty = \begin{pmatrix} p & p \\ 1-p & 1-p \end{pmatrix}$ where $p = \frac{b}{a+b}$

if α, β are not both 0, 1, when $n \to \infty$,

$1-\alpha_n-\beta_n := (1-\alpha-\beta)^n \longrightarrow 0$

$\therefore \alpha_n + \beta_n \to 1$



$\alpha_n + \beta_n = 1-(-1)^n$

$$P^n = \frac{1}{\alpha+\beta}\begin{pmatrix} \alpha\lambda^n+\beta & \beta-\beta\lambda^n \\ \alpha-\alpha\lambda^n & \alpha+\beta\lambda^n \end{pmatrix}$$

$\therefore \alpha_n = \frac{\alpha}{\alpha+\beta}(1-\lambda^n)$ $\quad (\lambda = 1-\alpha-\beta)$

$\beta_n = \frac{\beta}{\alpha+\beta}(1-\lambda^n)$

$\frac{\alpha_n}{\beta_n} = \frac{\alpha}{\beta} = \text{const}$.

when $n\to\infty$, $P^n = \frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \beta \\ \alpha & \alpha \end{pmatrix} = \frac{1}{\alpha+\beta}\begin{pmatrix} \beta \\ \alpha \end{pmatrix}(1\ 1)$

$P^n \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \frac{P_1+P_2}{\alpha+\beta}\begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \frac{1}{\alpha+\beta}\begin{pmatrix} \beta \\ \alpha \end{pmatrix}$

"Decimation" (1975 Kadanoff & Houghton)

Example 2 1d Ising Model.

$$H = -J\sum_i S_i S_{i+1}$$

$$Z = \mathrm{tr}\, e^{-\beta H} = \sum_{S_i=\pm1} e^{\beta J \sum_i S_i S_{i+1}} = \sum_{S_i=\pm1} \prod_i e^{\beta J S_i S_{i+1}}$$

$\cancel{\beta J}$ K

Now let's carry out the summation over odd sites.

$$\sum_{S_o=\pm1} \prod_i e^{K(S_{2i}S_{2i+1} + S_{2i+1}S_{2i+2})}$$

$$= \prod_i \sum_{S_{2i+1}=\pm1} e^{K(S_{2i}\ \cdots\cdots)}$$

$$= \prod_i 2\cosh(K(S_{2i}+S_{2i+2}))$$

$$= \prod_i e^{\ln 2(\cosh(J(S_{2i}+S_{2i+2})))}$$

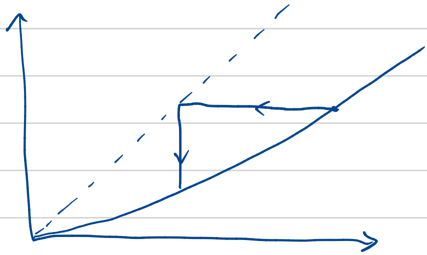
$$\therefore\ H_i' = -\ln 2(\cosh(J(S_{2i}+S_{2i+2}))) = \begin{cases} -\ln\cosh 2J - \ln 2 & \text{align} \\ -\ln 2 & \text{anti-align} \end{cases}$$

$$= -\frac{1}{2}\ln\cosh 2J\,(S_{2i}S_{2i+1}+1) - \ln 2$$

$$J' = \frac{1}{2}\ln\cos 2J\ . \quad < \frac{1}{2}\ln e^{2J} = J$$

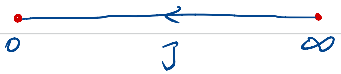
$K \ll 1$ $\quad \frac{1}{2}\ln\cosh K \approx 2K^2$

$K \gg 1$ $\quad \approx K - \frac{1}{2}\ln 2$



stable fixed point $K = 0$ (trivial)

un $K = \infty$



K is irrelevant near $K = \infty$

1d Ising model is disordered at finite temperature.

1.27 Scaling hypothesis

On the critical point, system is scale invariant.
Near · · systems are similar under scaling.

We expect the free energy per spin is a universal function
$$f(t,h) = -\log Z$$

After blocking $\overset{x \to x/L}{}$ $N \to N' = L^{-d}N$, $t \to t'$, $h \to h'$

$$N' f(t', h') = N f(t,h)$$

$$f(t,h) = L^{-d} f(t', h')$$

Near the critical point t is small, also assume h small
$t' = L^{y_t} t$, $h' = L^{y_h} h$ (y_t, y_h are scaling dim of t and h.)

$\therefore f(t,h) = L^{-d} f(L^{y_t} t, L^{y_h} h)$ (scaling hypothesis).

i.e. f is homogeneous. Therefore f can be expressed in terms of $\tilde{f}$: dimensionless $\mapsto$ dimensionless.
by dimensional analysis: $f = |t|^{d/y_t} \tilde{f}\left(\frac{h}{t^{y_h/y_t}}\right)$

proof: $f = \sum_b \lambda_b t^a h^b$
$$= L^{-d} \sum_b \lambda_b (L^{y_t} t)^a (L^{y_h} h)^b$$
$$= \sum_b L^{a y_t + b y_h - d} \lambda_b t^a h^b$$
$$\Rightarrow a y_t + b y_h = d$$
$$\therefore f = \sum_b \lambda_b t^{\frac{d - b y_h}{y_t}} h^b$$
$$= t^{\frac{d}{y_t}} \sum_b \lambda_b (t^{-\frac{y_h}{y_t}} h)^b.$$
$$\tilde{f}(x) = \sum_b \lambda_b x^b.$$

1.28

By dim analysis, $t \propto \xi^{-y_t}$

$\xi \propto t^{-\frac{1}{y_t}}$ $\qquad \frac{1}{y_t} := \nu$

$= t^{-\nu}$

$f \propto t^{d\nu}$

$\therefore$ specific heat $C = T^2 \frac{\partial^2}{\partial T^2} f \propto t^{d\nu - 2}$ (if $T_c \neq 0$)

$=: t^{-\alpha}$, where $\alpha = 2 - d\nu$. (hyperscaling relation)

order parameter $m = -\frac{\partial}{\partial h} f \propto |t|^{d\nu - \frac{y_h}{y_t}}$

$=: |t|^{\beta}$, where $\beta = d\nu - \frac{y_h}{y_t}$.

$\propto h^{\beta \frac{y_t}{y_h}} =: h^{\frac{1}{\delta}}$ where $\delta = \frac{y_h}{y_t} \cdot \frac{1}{\beta}$.

susceptibility $\chi = \frac{\partial m}{\partial h} \propto t^{d\nu - 2\frac{y_h}{y_t}}$

$=: t^{-\gamma}$ where $\gamma = 2\frac{y_h}{y_t} - d\nu$.

correlation function $\chi = \int d^d r \langle \sigma(0) \sigma(r) \rangle$

$\therefore \langle \sigma(0) \sigma(r) \rangle \propto t^{-\gamma} r^{-d}$

$\propto r^{\gamma y_t - d}$

$=: r^{-(d-2+\eta)}$ where $\eta = -y_t \gamma + 2$

deviation from Naïve dim (anomalous dim : $\frac{\eta}{2}$) $\qquad = d + 2 - 2y_h$.

disordered phase : α, γ, ν

ordered phase : α', β, γ', ν'

critical point : δ, η.

$\Delta_\sigma = \frac{1}{2}(d - 2 + \eta)$

MFT: $\begin{cases} \alpha = \alpha' = 0 \\ \beta = \frac{1}{2} \\ \gamma = \gamma' = 1 \\ \delta = 3 \end{cases}$ $\qquad \begin{cases} \nu = \frac{1}{2} \\ \eta = 0 \end{cases}$

1d Ising with source field.

$$Z = \sum_{\{\sigma_i\}} e^{\sum_i K_1 \sigma_i \sigma_{i+1} + \frac{1}{2} K_2 (\sigma_i + \sigma_{i+1})}$$

$$\begin{cases} K_1' = \frac{1}{4} \ln \left(\cosh(2K_1 + K_2) \cosh(2K_1 - K_2) \right) - \frac{1}{2} \ln \cosh K_2 \\ K_2' = K_2 + \frac{1}{2} \ln \left(\cosh(2K_1 + K_2) / \cosh(2K_1 - K_2) \right) \end{cases}$$

Let $g(x) = \frac{1}{2} \ln \cosh(x)$, then

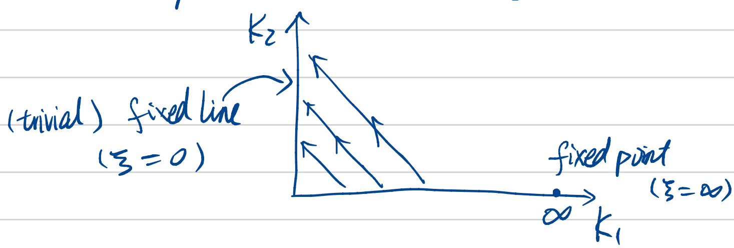
$$K_1' = \frac{1}{2}\left(g(K_2 + 2K_1) + g(K_2 - 2K_1)\right) - g(K_2)$$

Since g is convex, $K_1' \geq 0$

→ physics: antiferromagnet acts like ferromagnet

transformation is invariant under $K_2 \to -K_2$

→ physics: $\mathbb{Z}_2$ symmetry of zero-field Ising model.



Near $K_1 = \infty$, $K_2 = 0$,

$$\begin{cases} K_1' \approx K_1 - \frac{1}{2} \ln 2 \\ K_2' \approx 2K_2 \end{cases}$$

($t = \frac{T - T_c}{T_c}$ doesn't work since $T_c = 0$)

Change of variable $t = e^{-pK_1}$ $\quad (t \approx 0)$

$$\therefore t' \approx 2^{\frac{p}{2}} t$$

⇓ essential singularity

similar as KT transition

We see $y_1 = \frac{p}{2}$, $y_2 = 1$

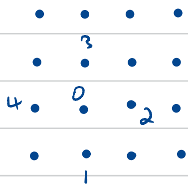
$$\therefore \begin{cases} \alpha = 2 - \frac{2}{p} \\ \beta = 0 \\ \gamma = \frac{2}{p} \\ \delta = \infty \end{cases} \qquad \begin{cases} \nu = \frac{2}{p} \\ \eta = 1 \end{cases}$$

Wilson did a brute force calculation, keeping 418 interaction parameters. The result were accurate and confirmed his hypothesis that the local couplings of shortest range were the most important.

11.1

Example 3 2d Ising Model.

Renormalization requirement:
system is bipartite, therefore we can sum over one part.



$$\sum_{s_0=\pm1} e^{K s_0(s_1+s_2+s_3+s_4)}$$
$$= 2\cosh K(s_1+s_2+s_3+s_4)$$

compare this with only K_1 $\begin{cases} 4 & \text{all align} \\ 0 & \text{1 anti} \\ 0 & \text{n.n.} \\ -4 & \text{n.n.n.} \end{cases}$ 2 anti

$$H' = -\ln 2\cosh K(s_1+\cdots+s_4) = \begin{cases} -\ln\cosh 4K & \text{all align} \\ -\ln\cosh 2K & \text{1 anti align} \\ 0 & 2 \end{cases}$$

What interactions can reproduce this one?
All terms respecting the original Z_2 symmetry:

quadratic: $\frac{1}{2}K_1'$ n.n., K_2' n.n.n.

quartic: $K_3' s_1 s_2 s_3 s_4$.

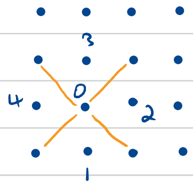
$$\begin{cases} 2K_1' + 2K_2' + K_3' = \ln\cosh 4K + C & \text{all align} \\ -J_3 = \ln\cosh 2J + C & \text{1 anti} \\ -2J_2 + J_3 = C & \text{2 anti n.n.} \end{cases}$$
$$-2K_1' + 2J_2 + J_3 = C \qquad \text{n.n.n.}$$

2020.1.29

Solution:
$$\begin{cases} K_1' = \frac{1}{4}\ln\cosh 4K_1 \\ K_2' = \frac{1}{8}\ln\cosh 4K_1 \\ K_3' = \frac{1}{8}\ln\cosh 4K_1 - \frac{1}{2}\ln\cosh 2K_1 \\ -C = \frac{1}{2}\ln\cosh 2K_1 + \frac{1}{8}\ln\cosh 4K_1 \end{cases}$$

Cut off at K_1 yields trivial result as 1d
Let's cut off at K_2.

but for K_2, summation only induce new interaction outside K_1 and K_2.

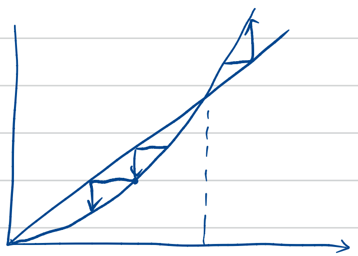


After a iteration, $K_2 \to K_1$, therefore

$$\begin{cases} K_1' = \frac{1}{4}\ln\cosh 4K_1 + K_2 \\ K_2' = \frac{1}{8}\ln\cosh 4K_1 \end{cases}$$

Solve $\begin{cases} K_{1c}^* = \frac{1}{4}\ln\cosh 4K_{1c}^* + K_{2c}^* \\ K_{2c}^* = \frac{1}{8}\ln\cosh 4K_{1c}^* \end{cases} \Rightarrow K_{1c}^* = \frac{3}{8}\ln\cosh 4K_{1c}^*$

$$\approx \begin{cases} 3K_1^2 & K_1 \ll 1 \\ \frac{3}{2}K_1 - \frac{3}{8}\ln 2 & K_1 \gg 1 \end{cases}$$



$K_{1c} = 0.507 \Rightarrow K_{2c} = \frac{1}{3}K_{1c} = 0.169$

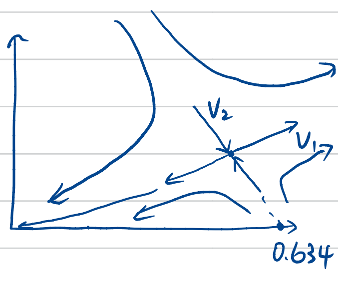
Linearize around $(0.507, 0.169)$,

$$\begin{cases} K_1' = 0.966\,K_1 + K_2 \\ K_2' = 0.483\,K_1 \end{cases} \quad \text{or} \quad \begin{pmatrix} K_1' \\ K_2' \end{pmatrix} = \begin{pmatrix} 0.966 & 1 \\ 0.483 & 0 \end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

Eigenvalues and eigenvectors

$\lambda_1 = 1.329 \qquad u_1 = \begin{pmatrix} 0.940 \\ 0.341 \end{pmatrix}$

$\lambda_2 = -0.363 \qquad u_2 = \begin{pmatrix} -0.601 \\ 0.799 \end{pmatrix}$



We have tune our original model to

$(0.63222320162791151, 0)$ to hit the critical point.

c.f. $K_c = \frac{1}{2}\sinh^{-1} 1 \approx 0.4407$.

If we had impliment renormalization for spin $\sigma' = \underline{\sqrt{2}^{\frac{\eta}{2}}}\sigma$, where $\eta = \frac{1}{4}$, famously claimed by Onsager (1948) and proved by Yang (1952).

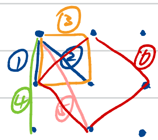
≈ 1.044.

Since $\xi = a t^{-\nu}$, $L^{-1}\xi = a(\lambda_1 t)^{-\nu}$

$\therefore L^{-1} = \lambda_1^{-\nu} \Rightarrow \nu = \frac{\ln L}{\ln \lambda_1} = 1.218$ c.f. $\nu = 1$.

Wilson: fixed points:

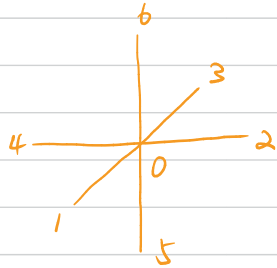
1. 0.281758
2. 0.095562
3. −0.017242
4. 0.008422
5. 0.004704
6. −0.004008



2.12

3d Ising Model :

$$H' = \ln\cosh K(\sigma_1 + \sigma_2 + \cdots + \sigma_6)$$



6 different configurations

↑ states : ① ϕ

② 1

③ 1 , 2

④ 1 , 3

⑤ 1 , 2 , 3

⑥ 1 , 2 , 5

			# of bonds
6 couplings :	$\frac{1}{4}K_1$	: 1, 2	12
	K_2	: 1, 3	3
	K_3	: 1, 2, 3, 4	3
	K_4	: 1, 2, 3, 5	12
	K_5	: 1, ..., 6	1
	C		1

Matching Hamiltonian.

$$\begin{array}{c} e \\ o \\ e \\ e \\ o \\ o \end{array} \begin{pmatrix} 3 & 3 & 3 & 12 & 1 & 1 \\ 1 & 1 & -1 & -4 & -1 & 1 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ -1 & 3 & 3 & -4 & 1 & 1 \\ -1 & 1 & -1 & 4 & -1 & 1 \\ 0 & -3 & 3 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} K_1' \\ K_2' \\ K_3' \\ K_4' \\ K_5' \\ K_6' \end{pmatrix} = \begin{pmatrix} \ln\cosh 6K_1 \\ \ln\cosh 4K_1 \\ \ln\cosh 2K_1 \\ \ln\cosh 2K_1 \\ 0 \\ 0 \end{pmatrix}$$

Solution:

$$K_1' = \frac{1}{8}(\ln\cosh 6K_1 + 2\ln\cosh 4K_1 - \ln\cosh 2K_1) + K_2$$
$$K_2' = \frac{1}{4}K_1'$$

$$\begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix} = \begin{pmatrix} 0.249311 \\ 0.049862 \end{pmatrix}$$

$$\begin{pmatrix} K_1' \\ K_2' \end{pmatrix} = \begin{pmatrix} 1.32348 & 1 \\ 0.09748 & 0 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

$$\lambda_1 = 1.39343 \qquad V_1 = \begin{pmatrix} 0.997562 \\ 0.0697852 \end{pmatrix}$$

$$\lambda_2 = -0.0699558 \qquad V_2 = \begin{pmatrix} -0.583048 \\ 0.812438 \end{pmatrix}$$

$$\Rightarrow K_{1c}(K_2 = 0) =$$

1.31 Limit cycle of RG in QM.

David Kaplan, Dam Son, etc. (2009)

Consider the potential $V(x) = \frac{\alpha}{x^2} - g\,\delta(x)$

The potential is singular at the origin, we introduce a UV cutoff $\Lambda_{UV} = x_0^{-1}$, and write the following potential.

$$V(r) = \begin{cases} \frac{\alpha}{x^2} & x > x_0 \\ -\frac{g}{x_0^2} & x < x_0 \end{cases}$$

We solve the Schrödinger eq. at $E \simeq 0$, boundary between bound state and scattering state.

$$\begin{cases} -\frac{1}{2m}\frac{d^2}{dx^2}\psi + \frac{\alpha}{x^2}\psi = 0 & |x| > x_0 \quad ① \\ -\frac{1}{2m}\frac{d^2}{dx^2}\psi - \frac{g}{x_0^2} = 0 & |x| < x_0 \quad ② \end{cases}$$

we see m just rescale α and g, let's set $2m = 1$

① : $\psi(x) = C_1 x^{S_1} + C_2 x^{S_2}$

② : $\tilde{\psi}(x) = C \cos\left(\sqrt{\frac{g}{x_0^2}}\, x\right)$

$\frac{d^2}{dx^2} x^S = S(S-1)x^{S-2} = \alpha x^{S-2}$

$\Rightarrow S_{1,2} = \frac{1}{2}(1 \mp \sqrt{1+4\alpha})$

match the boundary condition :

$$\psi(x_0) = \tilde{\psi}(x_0) \Rightarrow \quad C_1 x_0^{S_1} + C_2 x_0^{S_2} = C\cos\sqrt{g}$$

$$\psi'(x_0) = \tilde{\psi}(x_0) \Rightarrow \quad C_1 S_1 x_0^{S_1-1} + C_2 S_2 x_0^{S_2-1} = -C\frac{\sqrt{g}}{x_0}\sin\sqrt{g}$$

$$\therefore \frac{C_1 S_1 x_0^{S_1-1} + C_2 S_2 x_0^{S_2-1}}{C_1 x_0^{S_1} + C_2 x_0^{S_2}} = -\frac{\sqrt{g}}{x_0}\tan\sqrt{g} := -\frac{\gamma}{x_0}$$

$$\frac{S_1 + \left(\frac{C_2}{C_1}\right) S_2 x_0^{S_2-S_1}}{x_0 + \left(\frac{C_2}{C_1}\right) x_0^{S_2-S_1+1}} = -\frac{\gamma}{x_0}$$

$$\therefore \frac{C_2}{C_1} = -x_0^{S_1-S_2}\frac{\gamma+S_1}{\gamma+S_2}$$

Now we want to change the cutoff x_0 but leave the long-range wave function unchanged. This requires

$$0 = \frac{d}{dx_0}\left(\frac{C_2}{C_1}\right) = -(S_1 - S_2)\, x_0^{S_1-S_2-1}\,\frac{\gamma+S_1}{\gamma+S_2} - x_0^{S_1-S_2}\,\frac{S_2-S_1}{(\gamma+S_2)^2}\,\frac{\partial\gamma}{\partial x_0}$$

$$\therefore\ x_0\frac{\partial\gamma}{\partial x_0} = (\gamma+S_1)(\gamma+S_2)$$

$$\beta(\gamma) = \frac{\partial\gamma}{\partial t} = -(\gamma+S_1)(\gamma+S_2) \qquad t := -\ln x_0$$

$$= -\left(\gamma+\frac{S_1+S_2}{2}\right)^2 + \left(\frac{S_1-S_2}{2}\right)^2$$

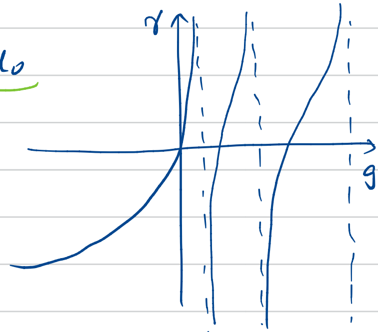
$$= -\left(\gamma+\frac{1}{2}\right)^2 + \alpha + \frac{1}{4}$$

When $\alpha < -\frac{1}{4}$, there is no solution, λ increase monotonically in $\mathbb{R}$.
The "time" (Log Energy) for γ flow from $-\infty$ to ∞ is

$$T = \int dt = \int_{-\infty}^{\infty}\frac{d\gamma}{\beta(\gamma)} = \int_{-\infty}^{\infty}\frac{d\gamma}{-(\gamma+\frac{1}{2})^2+\alpha+\frac{1}{4}} = \frac{\pi}{\sqrt{\frac{1}{4}-\alpha}}$$

$\Rightarrow$ There are scales $\underbrace{\mu_0,\ \lambda\mu_0,\ \cdots \lambda^n\mu_0}_{\text{Efimov state}}$

where $\lambda = e^{-\frac{\pi}{\sqrt{\frac{1}{4}-\alpha}}}$



Suppose at μ_0, $\gamma = -\infty$, $g = g_0$ is a zero of $\frac{1}{\gamma(g)}$.

Then at $\lambda\mu_0$, $\gamma = +\infty$, $g = g_1$ is the next zero of $\frac{1}{\gamma(g)}$, etc.

Therefore there is a limit cycle behavior in γ.

2.2 Kramers-Wannier duality

$$Z = \sum_{\sigma_i = \pm 1} \prod_{\langle i,j \rangle} e^{K\sigma_i\sigma_j}$$

High temperature (small K) expansion

$$e^{K\sigma_i\sigma_j} = \cosh K + \sigma_i\sigma_j \sinh K = \cosh K (1 + \sigma_i\sigma_j v)$$

$v = \tanh K$ is small.

Then $\prod_{\langle i,j \rangle} e^{K\sigma_i\sigma_j} = (\cosh K)^{2N} \prod_{\langle i,j \rangle} (1 + \sigma_i\sigma_j v)$ $\quad$ $2N$: # of bonds.

$$\therefore Z = (\cosh K)^{2N} \sum_{\sigma_i = \pm 1} 1 + v \sum_{\langle i,j \rangle} \sigma_i\sigma_j + v^2 \sum_{\substack{\langle i,j \rangle \langle k,l \rangle \\ \langle i,j \rangle \neq \langle k,l \rangle}} \sigma_i\sigma_j\sigma_k\sigma_l$$

if # of σ_i is odd, summation $\sum_{\sigma_i = \pm 1} = 0$

Therefore # of σ_i must be even $\Rightarrow$ closed loop

$$\therefore Z = 2^N (\cosh K)^{2N} \sum_r n(r) v^r$$

$n(r)$ = # of closed graphs with r bonds.

Low temperature expansion :

Ground state dominant, expand around ground state

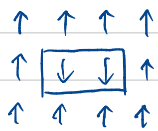
Count anti-aligned bonds : r

aligned bonds : $2N - r$

$$H = -J(2N - r - r) = -2J(N - r)$$

$$\therefore Z = \sum_r m(r) e^{2K(N-r)} = e^{2KN} \sum_r m(r) e^{-2Kr}$$

$m(r)$ = # of configs with r anti-aligned bonds.



Suppose we have a small K_0, then

$$Z_0 = 2^N (\cosh K_0)^{2N} \sum_r n(r) (\tanh K_0)^r$$

Now find $\tilde{K}_0$, s.t. $e^{-2\tilde{K}_0} = \tanh K_0 \Rightarrow \tilde{K}_0$ is large, then

$$\tilde{Z}_0 = e^{2\tilde{K}_0 N} \sum_r m(r) e^{-2\tilde{K}_0 r}$$

$$= e^{2\tilde{K}_0 N} \sum_r n(r) (\tanh K_0)^r$$

$$= (\tanh K_0)^{-N} \cdot 2^N (\cosh K_0)^{2N} Z_0$$

$$= 2^N (\cosh K_0 \sinh K_0)^N Z_0$$

$$= (\sinh 2K_0)^N Z_0$$

$\therefore$ K_0 and $\tilde{K}_0$ have same physics.

We know Z_c is diverge iff at T_c

if there is a unique T_c, then it must be self dual.

$$\Rightarrow \sinh 2K_c = 1 \quad \Rightarrow K_c = \frac{1}{2} \sinh^{-1} 1 .$$

2.2 ε-expansion