Einsteins Equations, Curvature, the Hilbert Action

So far we have argued that to understand free-falling objects we may characterize the paths that these objects take as "geodesic curves", $\gamma(s)$

$$\nabla_{\dot{\gamma}(s)} \dot{\gamma}(s) = 0$$

This leads to the notion of <u>connection</u>, ∇, or "Covariant Derivative"

as a means of differentiating vectors.

For the complete picture we also equip spacetime with a metric of lorentzian signature, g.

Specifying ∇ and g are then sufficient to describe free fall under gravity as geodesics.

<u>Today</u>

For simplicity we will restrict $\nabla = \bar{\nabla}$ the levi-civita connection, which can be determined uniquely from g

$$\Gamma^i_{jk} = \frac{1}{2} g^{is} \left(g_{sj,k} + g_{ks,j} - g_{jk,s} \right)$$

The natural question arises:

Which metrics are capable of describing our spacetime?

Looking Forward, we have an answer from Einstein

$$G + \Lambda g = 8\pi T$$

"Curvature" = "Stress energy"

From an action viewpoint this is the critical point of the $\underbrace{\text{Hilbert Action}}$

$$S_H = \int dV(R - 2\Lambda + \dots)$$

But what does all of this mean?

$G_{\mu\nu}$ = Einstein Curvature = $R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$ $\qquad$ $R_{\mu\nu}$ = Ricci Curvature

$T_{\mu\nu}$ = Stress Energy Tensor $\qquad$ R = Scalar Curvature

In fact, we will see that given a few assumptions, Einsteins Equation is the only Equation of its type and the Hilbert Action is the most general action under these assumptions.

First lets take a digression to better understand Curvature and stress energy.

Definition: Parallel Vector field.

Let $\gamma(s)$ be a curve in a smooth manifold M, equipped with metric g

A vector field X on M is said to be parallel to γ

(at least defined along γ)

if

$$\nabla_{\dot\gamma(s)} X \big|_{\gamma(s)} = 0$$

Definition: Parallel Transport

Let $\gamma(s)$ be a curve on M such that $\gamma(0) = p \in M$.

Let V_0 be some vector $\in T_pM$

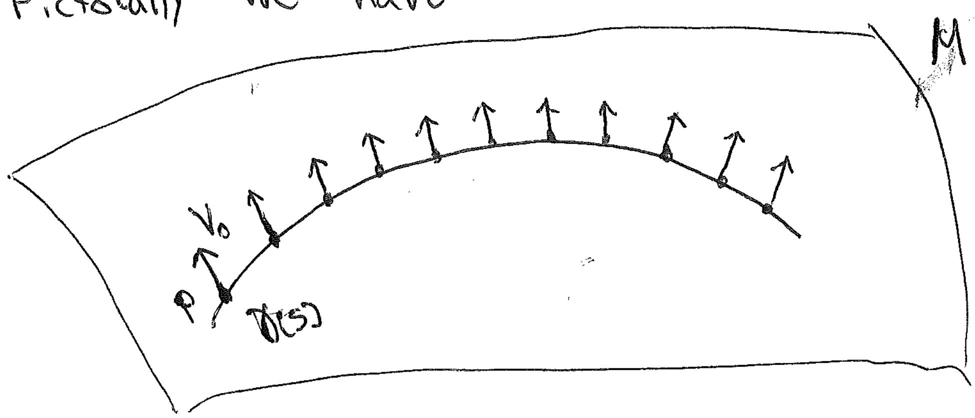
The parallel transport of V_0 is the unique vector field along γ such that

$$\nabla_{\dot\gamma(s)} V \big|_{\gamma(s)} = 0 \quad \text{and} \quad V\big|_{\gamma(0)} = V_0$$

In components

$$\dot\gamma^\mu \partial_\mu V^\beta = -\Gamma^\beta_{\mu\alpha} \dot\gamma^\mu V^\beta \qquad \text{just a first order ODE}$$

Pictorally we have



Actually, that picture is only right if γ is a geodesic of M.

See next page

The angle between $\dot{\gamma}$ and the parallel transport of $V_0 \in T_pM$ is only covariantly constant if γ is a geodesic

$$\frac{d}{ds}\langle \dot{\gamma}(s), V|_{\gamma(s)} \rangle = \langle \nabla_{\dot{\gamma}}\dot{\gamma}, V|_{\gamma} \rangle + \underbrace{\langle \dot{\gamma}, \nabla_{\dot{\gamma}} V|_{\gamma} \rangle}_{\text{zero since } V \text{ is parallel}}$$

$$= \underbrace{\langle \nabla_{\dot{\gamma}}\dot{\gamma}, V|_{\gamma} \rangle}_{\text{only zero if } \nabla_{\dot{\gamma}}\dot{\gamma} = 0}$$

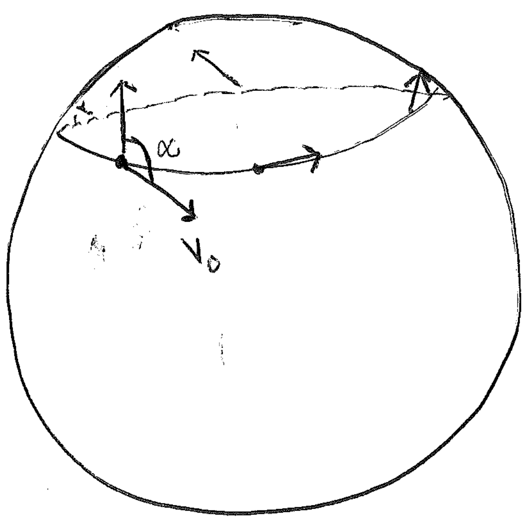
* Though, angles between two parallel transported vectors V and W are preserved

The magnitude of V_0 is preserved along γ, even if γ isn't a geodesic

$$\frac{d}{ds}\langle V|_{\gamma(s)}, V|_{\gamma(s)} \rangle$$

$$= 2\langle \nabla_{\dot{\gamma}(s)} V|_{\gamma(s)}, \dot{\gamma}(s) \rangle = 0$$

An angle "non-preserving" parallel vector field (non geodesic)



in fact we rotate by

$\alpha = 2\pi \cos\theta$

$\theta =$ latitude

To re-iterate:

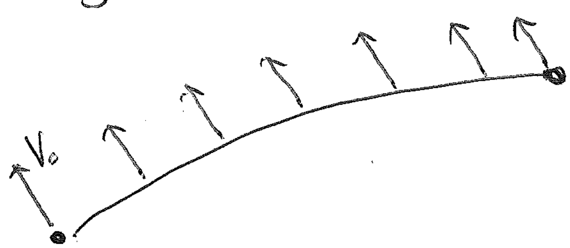
In flat space $\Gamma^{\alpha}_{\beta\gamma} = 0$ so the equations of parallel transport are trivial

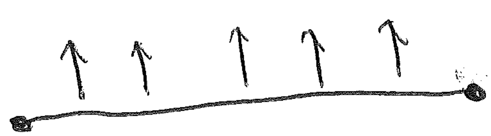
$$\dot{\gamma}^{\mu} \partial_{\mu} V^{\beta} = 0$$

The Vector does not change in any direction or along any curve

<u>ex</u>

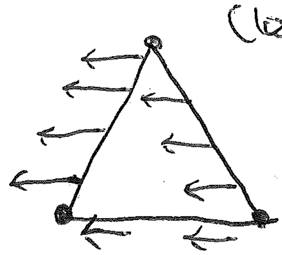
non-geodesic





geodesic

"Closed loop"



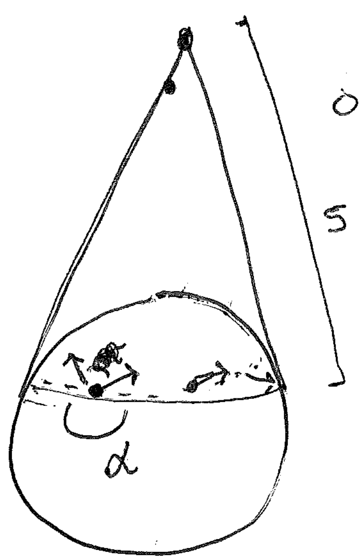
Parallel vector fields have vectors which are always parallel to each other in flat space

hence the name parallel

So what does parallel mean if not "parallel to the velocity of the curve"?

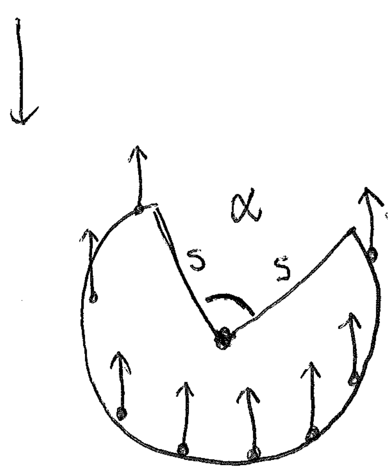
It means if we projected the vector field into a flat space with no curvature that all the lines of the field are parallel to one another. $\boxed{\text{Parallel transport in flat space keeps the vector itself constant in direction}}$

<u>ex revisited</u>



wrap a cone around a line of lattitude

Then cut it and unroll into a flat circle.



we see that when we re-wrap the cone that the vector must rotate by $(2\pi - \alpha)$

Equations of Geodesic Deviation

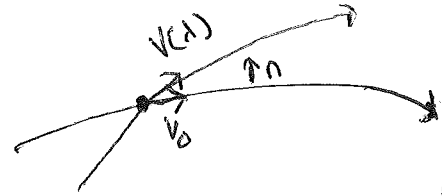
Consider the set of unit speed geodesics passing through $p \in M$ and parameterize it as

$\gamma_p(s, \lambda)$ where s denotes the position along the geodesic and λ is a smooth parameter which selects a unit vector $V(\lambda) \in T_pM$ such that $\dot{\gamma}_p(0,\lambda) = V(\lambda)$

that is $\gamma_p(0, \lambda) = p \;\; \forall \lambda$

(*) and $\dot{\gamma}_p(0,\lambda) = V(\lambda)$

denoting $V(\lambda = 0) = V_0$



(*) $\dot{\gamma} \equiv \dfrac{\partial \gamma}{\partial s}$

Definition

The <u>geodesic separation vector</u> n is defined as

$$n \equiv \frac{\partial \gamma_p}{\partial \lambda}(s, \lambda)$$

in a neighborhood of p we can understand this as

$n(\Delta\lambda) =$ " distance between geodesic $\gamma_p(s,\lambda)$ and geodesic $\gamma_p(s, \lambda + \Delta\lambda)$ "

We can ask how n changes along the direction of $\dot{\gamma}$, in which case we have a "relative velocity" of the geodesics

$$V_{rel} = \nabla_{\dot{\gamma}(s)} n$$

this of course depends on where we are at along the geodesic

We may more instructively want to know the "relative acceleration" of these nearby geodesics and ask about

$$a_{rel} = \nabla_{\dot\gamma(s)} \nabla_{\dot\gamma(s)} n$$

Now note a few things:

- $\nabla_{\dot\gamma} \dot\gamma = 0$ implies that $\nabla_n \nabla_{\dot\gamma} \dot\gamma = 0$
- using commutator rule we have

$$\nabla_{\dot\gamma} \nabla_n \dot\gamma + [\nabla_n, \nabla_{\dot\gamma}] \dot\gamma = 0$$

- then $\nabla_n \dot\gamma - \nabla_{\dot\gamma} n - [n, u] = 0$ <u>only if torsion free</u>

(but we have been assuming this anyways this lesson)

so $[n, u] = \frac{\partial}{\partial n} \frac{\partial}{\partial \lambda} - \frac{\partial}{\partial \lambda} \frac{\partial}{\partial n} = 0$

vanishes since partial derivatives commute

long story short $\nabla_{\dot\gamma} \nabla_{\dot\gamma} n + [\nabla_n, \nabla_{\dot\gamma}] \dot\gamma = 0$

which says $a_{rel} = -[\nabla_n, \nabla_{\dot\gamma}] \dot\gamma$

This inspires us to look for a tensor which describes such deviations at each point on M. The result will be the Riemann Curvature tensor

From the form of

$$a_{rel} = -[\nabla_n, \nabla_{\dot\gamma}]\dot\gamma$$

We want a tensor which takes in 3 vectors at a point and returns yet another vector. The natural choice is then a (1,3) tensor Riemann$(\bullet\ \bullet\ \bullet\ \bullet)$ such that

$$\text{Riemann}(\bullet, A, B, C) = [\nabla_A, \nabla_B]C$$

where the last slot is for a 1-form ω, or "dual vector"

$$\text{Riemann}(\omega, A, B, C) \equiv \omega([\nabla_A, \nabla_B]C) = \langle \omega^*, [\nabla_A, \nabla_B]C \rangle$$

This however <u>doesn't form a tensor</u> as we have written.

Primarily because it is not multiplicative linear in C

$$C' = f(p)C$$

$$[\nabla_A, \nabla_B]C' = C\nabla_{[A,B]}f + f[\nabla_A, \nabla_B]C$$

To form a tensor we can just "subtract" this difference term

<u>def</u>

Curvature operator: $R(A,B)C \equiv [\nabla_A, \nabla_B]C - \nabla_{[A,B]}C$

We then capture the curvature tensor as

$$\boxed{\text{Riemann}(\omega, A, B, C) = \langle R(A,B)C, \omega^* \rangle}$$

In components we write

$$\text{Riemann}(\omega^i, \partial_j, \partial_k, \partial_l) = R^i{}_{jkl}$$

And if we want we can consider the (0,4) tensor version as

$$\text{Riemann}(\partial_i, \partial_j, \partial_k, \partial_l) = \langle R(\partial_j, \partial_k)\partial_l, \partial_i \rangle$$

$$= R_{ijkl} = g_{im} R^m{}_{jkl}$$

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

Returning to the geodesic deviation we see that

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma} n + \text{Riemann}(\cdot, \dot\gamma, n, \dot\gamma) = 0$$

We may also define the Ricci tensor as

$$\text{Ricci}(A,B) = \text{Riemann}(A, \partial_i, B, \partial_i)$$

$$R_{ij} \Rightarrow R^l{}_{ilj} = R_{ij}$$

And the <u>Scalar</u> curvature R as

$$R = \text{Ricci}(\partial_i, \partial_i)$$

$$R = R^{l}{}_{il}{}^{i} = R_{ij} g^{ij}$$