Test II
PHY176
Spring, 2008

Rules: No notes, books or other materials permitted

1. (20 points) We used the fact that photons have frequencies \( \omega = c k \) to compute the energy of a black body. Here, \( k = n(\pi/L) \), where \( L \) is the system length (for a cube). Here, \( n^2 = n_x^2 + n_y^2 = n_z^2 \), where all the \( n_i > 0 \), as usual.

Now suppose in another universe that photons have frequencies \( \omega = ak^2 \), where \( a \) is some constant, and the \( k \)'s are the same.

a. In this new universe, how does the energy of a black body depend on \( \tau \)? (Do not worry about dimensionless integrals.) You should not have \( \beta \) or \( \tau \) inside an integral.

b. To what other system that we have studied can this result be applied?

2. (10 points) Suppose that the energy, \( \varepsilon_i \), of every particle is increased by an amount \( q\Delta V \), corresponding to the presence of an electric potential.

Use the Canonical Ensemble approach to show that the chemical potential consists of two parts: \( \mu = \mu_{\text{int}} + \mu_{\text{ext}} \), where the first part corresponds to the case where the potential is zero, and \( \mu_{\text{ext}} = q\Delta V \).

3. (20 points) In the book, there is an example of a very special Fermi system consisting of only one state of energy \( \varepsilon \), which can be occupied or not. This makes the summation of the Grand Canonical Partition Function very simple, and it is:

\[
Z = 1 + \exp[-\beta(\varepsilon - \mu)].
\]

a. Find expressions for the mean particle number, \( N \), and the mean energy, \( U \).

b. Find the GC Potential, \( \psi \) and the entropy, \( \sigma \).

c. Find the chemical potential as a function of \( N \) and \( \tau \).

d. Find the Helmholtz Free Energy, \( F \), as a function of \( N \) and \( \tau \). (Hint—note that \( \psi \) and \( F \) are related by Legendre transforms involving \( N \) and \( \mu \).)
Test II Solutions

1) \[ u = \frac{\sum k w_k}{k} e^{-\beta k w_k/2} - 1 \]

\[ = 2 \pi \sum \frac{ka (T/2)^2}{8} \int_0^\infty e^{-\beta ka (T/2)^2 n^2} n^2 \, dn \cdot n^2 \]

Let \( x = \beta ka (T/2)^2 n^2 = C n^2 \).

\[ n = \frac{1}{C (x/C)^2} \quad dn = \frac{1}{2} \left( \frac{1}{xC} \right)^2 \, dx \]

\[ C = \beta ka \left( \frac{T}{4} \right)^2 \]

\[ u = \pi \tau e C \int_{-\infty}^\infty \frac{x^2}{C^2} \, \frac{1}{e^x - 1} \, dx \]

\[ = \frac{\pi \tau C}{2} \int_0^\infty \frac{x^{3/2} \, dx}{e^x - 1} \]

But now \( C = \frac{ka (T/2)^2}{C} \)

So \( u = \frac{\pi}{2} \tau \sqrt{2} \left( \frac{\beta}{\pi} \right) \frac{1}{ha} \frac{1}{r^2} \]

\[ = \frac{\pi}{2} \tau \sqrt{2} L^3 \left( \frac{4}{h a} \right)^{-3/2} \]
Note that this is what one would expect for \( B \cdot E \) below \( \tau \).

\[ e' = e + N\gamma \Delta V \]
\[ z' = \sum e - \beta (e + N\gamma \Delta V) \]
\[ = \sum e - \beta e \gamma \Delta V \sum e - \beta e \]
\[ = e - \beta \gamma \Delta V \cdot z \]

\[ F' = -2 \log z' = -2 \left[ -\beta \gamma \Delta V \cdot N + \log z \right] \]

\[ = \beta \gamma \Delta V \cdot N + F \Rightarrow \]

\[ \mu' = \mu + g \Delta V \]
\[ \Delta V = 0. \]
\[ N = \left( \frac{\partial z}{\partial \beta \mu} \right) \beta = \frac{1}{1 + e^{-\beta (E - \mu)}} \leq 1 = \frac{\beta (E - \mu)}{e^{\beta (E - \mu)} + 1} \]

\[ u = \frac{\partial \langle z \rangle}{\partial \beta \mu} \beta \mu = \frac{-E + \beta (E - \mu)}{1 + e^{-\beta (E - \mu)}} = \frac{e}{e^{\beta (E - \mu)} + 1} \]

\[ 4 = -\tau \log z = -\tau \log \left[ \frac{1}{\tau} + \frac{\exp[-\beta (E - \mu)]}{1 + \exp[-\beta (E - \mu)]} \right] \]

\[ v = \frac{\partial \langle y \rangle}{\partial \mu} = \log \left[ \frac{1}{\tau} + \frac{\exp[-\beta (E - \mu)]}{1 + \exp[-\beta (E - \mu)]} \right] = \log \left[ \frac{\exp[-\beta (E - \mu)]}{\beta (E - \mu)} \right] + \frac{1}{\beta (E - \mu)} \]

\[ \frac{1}{n} = 1 + e^{-\beta (E - \mu)} \]
So \( \frac{1}{N} - 1 = \frac{1 - N}{N} = e^{\beta(\varepsilon - \mu)} \)

\[ \varepsilon - \mu = \tau \log \left( \frac{1 - N}{N} \right) \]

\[ \mu = \varepsilon - \frac{\tau}{2} \log \left( \frac{1 - N}{N} \right) \]

Then

\[ F = u - 2\sigma \]

\[ y = u - 2\sigma - \mu N \]

\[ \Rightarrow \]

\[ F = y + \mu N \]

Note that

\[ e^{-\beta(\varepsilon - \mu)} = \frac{N}{1 - N} \]

\[ y = -\tau \log \left[ 1 + \exp \left[ -\beta(\varepsilon - \mu) \right] \right] \]

\[ = -\tau \log \left[ 1 + \frac{N}{1 - N} \right] \]

\[ = -\tau \log \left[ \frac{1 - N + N}{1 - N} \right] = \tau \log (1 - N) \]

So

\[ F(\tau, N) = \tau \log (1 - N) + \]

\[ N (\tau - 3 - \tau \log \left( \frac{1 - N}{N} \right)) \]