2) Since \( e \approx p e c \) and

\[
P = (h\pi (L))^2 \quad \text{and} \quad N^2 = 4h^2 \pi^2 \quad \text{and} \quad 4h^2 \pi^2 + n^2 \]

we have as before that

\[
N = 2 \cdot \frac{1}{8} \frac{2}{3} \pi^3 N_F^3 = \frac{\pi}{6} n_F^3
\]

Hence \( n_F = (8N/\pi)^{1/3} \) and

\[
\varepsilon_F = \frac{\gamma n_F}{c}
\]

\[
= \kappa \pi c \left( \frac{3N/\pi}{2} \right)^{1/3} \n_F^{1/3}
\]

\[
= \kappa \pi c \left( 3N/\pi \right)^{1/3} \n_F^{1/3} \quad \text{h} \quad n = N/V
\]

b) \( u_0 = 2 \cdot \frac{1}{8} \cdot \frac{2}{3} \pi^3 \cdot \int_{-\pi/2}^{\pi/2} \varepsilon(n) \pi n \sin \theta \, dn \)

\[
= \frac{\pi}{6} \cdot \int_{0}^{2\pi} \frac{k\pi^2 c}{L} \cdot n^3 \, dn
\]

\[
= \frac{k\pi^2 c}{V^{1/3}} \cdot \frac{1}{6} \cdot \frac{3N}{\pi} \cdot n_F
\]
Problem Ch 7
-2-

\[ \frac{3kT}{4} \sum_{n} N = \frac{3}{4} \varepsilon \cdot N \]

3) Note that \( du = \partial \alpha - pdV + qdN \)

Hence, at \( \varepsilon = 0 \) and \( N = \text{const} \)

\[ du = -pdV \]

For \( du \) \( \varepsilon = 0 \) ground state

\[ u = u_0 = \frac{3}{2} N \varepsilon \]

\[ \varepsilon = \frac{k^2}{2m} \left( \frac{3\pi^2 N}{\varepsilon} \right)^{2/3} \]

Hence, for \( \varepsilon = 0 \)

\[ P = \left( \frac{\partial u}{\partial V} \right)_N = -\frac{3}{2} N \left( \frac{\partial \varepsilon}{\partial N} \right)_N \]

\[ = -\frac{3}{2} \cdot N \cdot \left( -\frac{2}{3} \varepsilon / V \right) \]

\[ = \frac{2}{3} N \varepsilon / V \]

\[ = \frac{2}{3} \cdot \frac{N}{V} \cdot \frac{k^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3} \]

\[ = \frac{(3\pi^2)^{2/3} k^2}{5} \frac{m}{N} \left( \frac{N}{V} \right)^{5/3} \]
b) Note that

\[ C_V = \frac{1}{3} \pi^2 D(\varepsilon_F) \varepsilon + \ldots \]

Hence \( \left( \frac{\partial S}{\partial T} \right)_V = \frac{1}{3} \pi^2 D(\varepsilon_F) \varepsilon + \ldots \)


and \( S = \frac{1}{3} \pi^2 D(\varepsilon_F) \varepsilon + \ldots \)

Note that since \( S (T=0) = 0 \), the integration constant is zero.

s) For \( ^3\text{He} \), \( \rho = 0.081 \text{ g/cm}^3 \)

\[ \varepsilon_F = \frac{\hbar^2}{2m} \left( \frac{3 \pi^2 N}{4} \right)^{1/3} \]

We get \( m \) from \( M = \text{molecular weight} = 3.0 \text{ g/mole} \)

\[ m = \frac{M}{6.0 \times 10^{-23} \text{ mole/dm}^3} = 5.0 \times 10^{-27} \text{ kg/mole/atom} \]

\[ N/V = \text{mole/atom} = \frac{\text{mole}}{\text{atom}} \]

\[ = 6.0 \times 10^{23} \text{ mole/atom} \times \frac{1}{3} \text{ g/mole} \times \frac{0.081 \text{ g/cm}^3}{10^{-6} \text{ cm}^3} \times \frac{10^6 \text{ cm}^3}{1 \text{ m}^3} = 1.6 \times 10^{28} \text{ mole/m}^3 \]

\[ \varepsilon_F = 6.7 \times 10^{-23} \frac{\text{J}}{\text{mol}} = k_B T_F = 1.4 \times 10^{-23} \frac{\text{J}}{\text{K}} \]

Hence \( T_F = 4.8 \text{ K} \)

\[ \varepsilon_F = \frac{1}{2} m u_F^2 \rightarrow u_F = 1.6 \times 10^2 \text{ m/s} \]

\[ C_V = \frac{1}{3} \pi^2 D(\varepsilon_F) \varepsilon \]

or \( \frac{C_V}{k_B} = 1.0 \text{ N}^2 / \text{K} \text{(likely misprint)} \)
8) For a Bose gas, for \( T < T_E \), \( \mu = 0 \) and \( U \) has a contribution only for those states with \( \varepsilon > 0 \). Hence we can write

\[
U = \int_0^\infty \varepsilon D(\varepsilon) f(\varepsilon) \, d\varepsilon
\]

\[
f = \frac{1}{e^{\beta \varepsilon} - 1} = \frac{1}{e^{\beta E} - 1} \quad (\text{Since } \mu = 0)
\]

and

\[
D(\varepsilon) = \frac{\sqrt{\frac{2M}{\hbar^2}}}{4\pi^2} \varepsilon^{3/2} = CV \varepsilon^{3/2}
\]

\[
(C = \frac{1}{4\pi^2} \left( \frac{2M}{\hbar^2} \right)^{3/2})
\]

Thus for \( T < T_E \), the integral is not too bad:

\[
U = CV \int_0^\infty \frac{\varepsilon^{3/2} \, d\varepsilon}{e^{\beta \varepsilon} - 1} = CV \cdot \frac{\varepsilon^{3/2} \, d\varepsilon}{e^{\beta \varepsilon} - 1}
\]

\[
\text{evaluated at } \beta T = \frac{1}{2}
\]

\( g^{3/2}(1) \) is a constant - see below where \( g^{3/2} \) crops up again.

We now obtain \( CV \) very simply:

\[
CV = \left( \frac{\partial U}{\partial T} \right)_V = \frac{5}{2} CV \varepsilon^{3/2}
\]
For \( \tau > \nu \varepsilon \) we have the usual expressions for \( N \) and \( U \):

\[
N = \int_0^\infty D(\varepsilon) f(\varepsilon) d\varepsilon = CV \int_0^\infty \frac{\varepsilon e^{\frac{\varepsilon}{2}}}{(BE-\varepsilon)^{1-\lambda}} \varepsilon^{\frac{\varepsilon}{2}} d\varepsilon - 1
= CV \int_0^\infty \frac{\varepsilon e^{\frac{\varepsilon}{2}}}{(1-\lambda \varepsilon) e^{\frac{\varepsilon}{2}}} \varepsilon^{\frac{\varepsilon}{2}} d\varepsilon
\]

\( U \) is quite similar to \( N \), with just an extra \( \varepsilon \) in the integrand:

\[
U = \int_0^\infty E(\varepsilon) f(\varepsilon) d\varepsilon = CV \int_0^\infty \frac{\varepsilon^{3/2} e^{\frac{\varepsilon}{2}}}{1-\lambda \varepsilon} \varepsilon^{\frac{\varepsilon}{2}} d\varepsilon
\]

To get \( CV \), we need \( CV = \frac{\partial U}{\partial \nu} \mid_{\nu=N} \). But we have \( U \) as a function of \( \nu \) and \( \lambda \) and \( \nu \) as independent variables (\( \nu \) goes along for the ride). Thus

\[
CV = \frac{\partial U}{\partial \nu} \mid_{\nu=N} = \frac{\partial (U \nu)}{\partial \nu} \mid_{\nu=N} = \frac{2U \nu}{\partial (2\nu)} \mid_{\nu=N} \frac{2U (2\nu / \nu)}{2} = \frac{2U (2\nu)}{\nu}
\]

\[
\frac{\partial (2\nu / \nu)}{\partial (2\nu)} = \frac{\partial (2\nu)}{\partial \nu} \frac{\partial \nu}{\partial \nu} = \left( \frac{2\nu}{\nu} \right)^2 = \left( \frac{2\nu}{\nu} \right)^2
\]
So to get \( CV \) for \( x > x_e \), we need both \( U(x, y) \) and \( N(x, y) \).

We can evaluate the integrals formally by

\[
N = CV \int_0^\infty \frac{e^{-x_e y} e^{-y}}{1 - y e^{-x_e y}} \, dy = CV \int_0^{\infty} \frac{dx}{1 - x e^{-x_e}} \int_0^\infty \frac{dx}{x} \int_0^\infty \frac{dx}{1 - \lambda e^{-x}}
\]

That is, we can expand \( \frac{1}{1 - \lambda e^{-x}} \) as a geometric series:

\[
\frac{1}{1 - y} = 1 + y + y^2 + \ldots
\]

We can now write

\[
N = CV \int_0^{\infty} \frac{dx}{x} \sum_{n=1}^\infty \frac{1}{n} x_n \quad \text{Note that} \quad \int_0^\infty \frac{dx}{x^m} = -\ln x
\]

Thus

\[
N = CA_{3/2} 2\pi \sum_{n=1}^\infty \frac{1}{n^{3/2}} \quad \text{Note that} \quad \int_0^\infty \frac{e^{-x}}{x^{3/2}} = \frac{\Gamma(1/2)}{2^{1/2}}
\]
In the same vein, \[ g_{\frac{1}{2}}(\lambda) \]

\[ U = CV A_{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\lambda^n}{n^{3/2}} \]

\[ A_{\frac{1}{2}} = \int_{0}^{\infty} dy y^{3/2} e^{-y} \]

We can now fill in terms like

\[ \left( \frac{\partial U}{\partial \lambda} \right)_{\lambda} = \frac{1}{2} CV A_{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{n \lambda^{n-1}}{n^{3/2}} \]

\[ \left( \frac{\partial U}{\partial \lambda} \right)_{\lambda} = CV A_{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{n^{3/2}} g_{\frac{1}{2}}(\lambda) \]

and similarly for the derivatives of \( N \).

To proceed further requires numerical calculation.

Combining with our result for \( CV \) when \( \lambda < \lambda_0 \), we get \( CV(\lambda) \) that looks like:

![Graph](image)
Problems Ch. 7 - 8 -

Problems 11 and 12

For one or initial

$$\langle AN^2 \rangle = \frac{\partial}{\partial \beta^m} \langle N \rangle = \frac{\partial}{\partial \beta^m} \frac{1}{e^{\beta(e-\mu)} + 1} = \frac{\beta(e-\mu)}{\left[ e^{\beta(e-\mu)} + 1 \right]^2}$$

($+$ = Fermi, $-$ = Bose)

$$= \frac{1}{e^{\beta(e-\mu)} + 1}$$

Note that

$$\frac{X}{x \pm 1} = \frac{x \pm 1 \mp 1}{x \pm 1} = \frac{1 \mp \frac{1}{x \pm 1}}$$

Hence,

$$\langle (AN)^2 \rangle = \left( \frac{1}{x \pm 1} \right) \left( 1 \mp \frac{1}{x \pm 1} \right)$$

$$= \langle N \rangle \left( 1 \mp \langle N \rangle \right)$$